Computing multiplier ideals in Macaulay2

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Analysis

- For a polynomial f, the function $|f|^\lambda$ is locally integrable $\operatorname{Re}\lambda > 0$.
- Hence, $|f|^\lambda$ is a generalized function defined on $\{\lambda : \operatorname{Re}\lambda > 0\} \subset \mathbb{C}$ that depends analytically on λ.
- Gelfand [1957]: Does it extend to a meromorphic function on $\mathbb{C}$?
- I. N. Bernstein [1968]: Yes. The poles are contained in a finite number of arithmetic progressions.
- Key ingredients: resolution of singularities and being able to write a functional equation

$$b(s) f^s = P \cdot f^{s+1}$$

when f is a monomial. (Here: $b(s)$ is a univariate polynomial and P is a linear differential operator with coefficients in $\mathbb{C}[\boldsymbol{x}, s]$.)

Invariants in singularity theory

Definition (Multiplier ideal for $\boldsymbol{f} = (f_1, \ldots, f_r)$)

$$\mathcal{J}(\boldsymbol{f}^c) = \left\{ h \in \mathbb{C}[\boldsymbol{x}] : \frac{|h|^2}{(\sum |f_i|^2)^c} \text{ is locally integrable } \right\}.$$

For $r = 1$, it is the ideal of h, that make $\frac{|h|}{|f_1|^c}$ locally integrable.

- Algebrao-geometric definition: via log-canonical resolutions.
- Jumping coefficients of $\boldsymbol{f}$: rational numbers

$$0 = \xi_0 < \xi_1 < \xi_2 < \cdots$$

  such that $\mathcal{J}(\boldsymbol{f}^c)$ is constant exactly for $c \in [\xi_i, \xi_{i+1})$.
- ξ_1 is called the log-canonical threshold.
- These invariants measure singularities of the corresponding variety; in particular, they depend only on the ideal $\langle \boldsymbol{f} \rangle$.

Weyl algebra

- Let K be a field of characteristic zero. (Think: $K = \mathbb{C}$)
- Affine space: $X = K^n$.
- Weyl algebra: an associative algebra

$$D_X = K\langle \boldsymbol{x}, \boldsymbol{\partial} \rangle = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

  where $[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1$ and all other pairs of generators commute.
- D_X is isomorphic to the algebra of linear differential operators with polynomial coefficients.
- Every element has the normal form

$$Q = \sum_{\alpha, \beta \in \mathbb{Z}^n} c_{\alpha\beta} \boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta,$$

  where finitely many of $c_{\alpha\beta} \in K$ are nonzero.

D-modules

- D_X is simple: only trivial two-sided ideals.
- Today we consider only left ideals and left D_X-modules.
- Examples of D-modules: $K[\boldsymbol{x}]$, $K[[\boldsymbol{x}]]$, $C^\infty(X)$.
- Another example: localization $K[\boldsymbol{x}, f^{-1}]$ where f is a nonzero polynomial:

$$x_i \cdot gf^{-j} = x_i g f^{-j},$$
$$\partial_i \cdot gf^{-j} = \left(\frac{\partial g}{\partial x_i} f - jg \right) f^{-j-1},$$

  for $1 \leq i \leq n$, $g \in K[\boldsymbol{x}]$, and $j \in \mathbb{Z}$.
- Software:
  - `kan/sm1` (Takayama)
  - `risa/asir` (Noro)
  - `dmod.lib, Singular` (Levandovsky et al.)
  - `D-modules, Macaulay2` (L., Tsai)

Gröbner bases

- D_X is Gröbner-friendly: D_X is an algebra of solvable type.
- Gröbner bases can be computed with respect to any w-compatible monomial order, where $w = (w_{\boldsymbol{x}}, w_{\boldsymbol{\partial}}) \in \mathbb{R}^{2n}$ satisfies $w_{\boldsymbol{x}} + w_{\boldsymbol{\partial}} \geq 0$ componentwise.

Several ways to define dimension of an ideal I:

- Gelfand-Kirillov dimension;
- dimension of the initial ideal $\mathrm{in}_w(I) \subset \mathrm{gr}_w D_X$, where the latter is a polynomial ring in $2n$ variables when $w_{\boldsymbol{x}} + w_{\boldsymbol{\partial}} > 0$.

Theorem (Fundamental theorem of algebraic analysis)

Let I be a nonzero left ideal in D_X, then $n \leq \dim I \leq 2n$,

Holonomic D-modules

- An ideal (or a D-module) is called holonomic if its dimension equals n.
- Examples:
  - any nontrivial principal ideal in case $r = 1$;
  - D_X-module $K[\boldsymbol{x}]$;
  - D_X-module $K[\boldsymbol{\partial}]$;
  - localization $K[\boldsymbol{x}, f^{-1}]$.
- Theorem: Every holonomic D-module is cyclic.
- Every holonomic $M = D_X \xi$ can be thought of as

$$M \cong D_X / \operatorname{Ann}_{D_X}(\xi).$$

Back to the problem...

- Recall: $D_X = K\langle \boldsymbol{x}, \boldsymbol{\partial} \rangle$ the Weyl algebra on $X = K^n$.
- $D_Y = K\langle \boldsymbol{x}, \boldsymbol{\partial}_{\boldsymbol{x}}, t, \partial_t \rangle$, the Weyl algebra on $Y = X \times K$.
- $V^\bullet D_Y$ is the V-filtration of D_Y along X:

$$V^m D_Y = D_X \cdot \{t^\mu \partial_t^\nu \mid \mu - \nu \geq m\}.$$

- V-filtration introduced by Kashiwara and Malgrange in 1980's.

Global Bernstein–Sato polynomial

- Action of D_Y on $N_f := K[\boldsymbol{x}][f^{-1}, s]f^s$:
  - x_i and ∂_{x_i} act naturally for $i = 1, \dots, n$;
  - for $h \in K[\boldsymbol{x}][f^{-1}, s]$,

$$t \cdot h(\boldsymbol{x}, s)f^s = h(\boldsymbol{x}, s+1)ff^s,$$
$$\partial_t \cdot h(\boldsymbol{x}, s)f^s = -sh(\boldsymbol{x}, s-1)f^{-1}f^s.$$

- For $f \in K[\boldsymbol{x}]$, the global Bernstein–Sato polynomial b_f of f, is the monic polynomial $b(s) \in K[s]$ of minimal degree such that

$$b(s)f^s = Pff^s$$

  for some $P \in D_X\langle -\partial_t t\rangle$.
- Alternatively, let $M_f := K[\boldsymbol{x}] \otimes_K K\langle\partial_t\rangle$ with actions of a vector field ξ on X and t:

$$\xi(p \otimes \partial_t^\nu) = \xi p \otimes \partial_t^\nu - (\xi f)p \otimes \partial_t^{\nu+1}$$
$$t \cdot (p \otimes \partial_t^\nu) = fp \otimes \partial_t^\nu - \nu p \otimes \partial_t^{\nu-1}.$$

- b_f = minimal polynomial of the action of $\sigma = -\partial_t t$ on $(V^0 D_Y)\delta/(V^1 D_Y)\delta$ with $\delta = 1 \otimes 1 \in M_f$.

Computing Bernstein–Sato polynomial

The first general algorithm is due to T. Oaku (1997).

- Let $I_f = \left\langle t - f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \right\rangle$.
- For the weight vector $w = (\mathbf{0}, 1) \in \mathbb{R}^n \times \mathbb{R}$, compute $\mathrm{in}_{(-w,w)} I_f$ via Gröbner bases.
- $\langle b_f(\sigma) \rangle = \mathrm{in}_{(-w,w)} I_f \cap K[\sigma]$. ← expensive elimination step

A shortcut (first used by Noro):

1. Let G be a Gröbner basis of $\mathrm{in}_{(-w,w)} I_f$;
2. Find the smallest d such that the normal forms $\mathrm{NF}_G(\sigma^i)$ for $0 \leq i \leq d$ are K-linearly dependent;
3. If $\mathrm{NF}_G(\sigma^d) + \Sigma_{i=0}^{d-1} c_i \, \mathrm{NF}_G(\sigma^i) = 0$ then $b_f(s) = s^d + \Sigma_{i=0}^{d-1} c_i s^i$.

Related work

- global b-function algorithms:
  - Briancon–Maisonobe alternative (Castro–Ucha, Levandovskyy et al.)
  - modular methods (Noro)
- local b-function (Oaku, Nakayama, Nishiyama–Noro);
- b-ideal (Bahloul–Oaku).

Bernstein–Sato polynomial for an arbitrary variety

- Let $\boldsymbol{f} = f_1, \dots, f_r \in K[\boldsymbol{x}]$, $\boldsymbol{f}^{\boldsymbol{s}} = \prod_{i=1}^r f_i^{s_i}$, and $Y = K^n \times K^r$ with coordinates $(\boldsymbol{x}, \boldsymbol{t})$.
- Action of $D_Y = K\langle \boldsymbol{x}, \boldsymbol{t}, \boldsymbol{\partial_x}, \boldsymbol{\partial_t}\rangle$ on $N_f := K[\boldsymbol{x}][\boldsymbol{f}^{-1}, \boldsymbol{s}]\boldsymbol{f}^{\boldsymbol{s}}$ generalizes the action in the case $r = 1$.
- Let $\sigma = -\left(\sum_{i=1}^r \partial_{t_i} t_i\right)$.
- The generalized Bernstein–Sato polynomial $b_{\boldsymbol{f},g}$ of $\boldsymbol{f}$ at $g \in K[\boldsymbol{x}]$ is the monic polynomial $b \in \mathbb{C}[\boldsymbol{s}]$ of the lowest degree for which there exist $P_k \in D_X\langle \partial_{t_i} t_j \mid 1 \le i, j \le r\rangle$ for $k = 1, \dots, r$ such that

$$b(\sigma) g \boldsymbol{f}^{\boldsymbol{s}} = \sum_{k=1}^r P_k g f_k \boldsymbol{f}^{\boldsymbol{s}}.$$

- When $r = 1$, we have $b_{\boldsymbol{f},1} = b_{f_1}$.

Connection between multiplier ideals and b-function

- The log-canonical threshold c_0 is the lowest root of $b_{\boldsymbol{f}}(-s)$.
- Every jumping coefficient $c \in [c_0, c_0 + 1)$ is a root of $b_{\boldsymbol{f}}(-s)$.
- The following provides a membership test.

Theorem (Budur–Mustaţă–Saito)

Let $g \in K[\boldsymbol{x}]$ and fix a positive rational number c. Then

$$g \in \mathcal{J}(\boldsymbol{f}^c) \Leftrightarrow c < \text{roots of } b_{\boldsymbol{f},g}(-s).$$

- Let $M_{\boldsymbol{f}} \cong K[\boldsymbol{x}] \otimes_K K\langle \boldsymbol{\partial_t} \rangle$ with the action of D_Y (similar to that on M_f); $M_{\boldsymbol{f}}$ is equipped with the V-filtration.
- Let $\delta = 1 \otimes 1 \in M_{\boldsymbol{f}}$, consider:

$$\overline{M}_{\boldsymbol{f}}^{(m)} = (V^0 D_Y)\delta / (V^m D_Y)\delta.$$

- The m-generalized Bernstein–Sato polynomial $b_{f,g}^{(m)}$ is the monic minimal polynomial of the action of σ on $(V^0 D_Y)\overline{g \otimes 1} \subseteq \overline{M}_{\boldsymbol{f}}^{(m)}$.
- $b_{\boldsymbol{f},g}^{(m)}$ is equal to the monic polynomial $b(s)$ of minimal degree such that there exist $P_k \in D_X\langle -\partial_{t_i} t_j \mid 1 \leq i,j \leq r \rangle$ and $h_k \in \langle \boldsymbol{f} \rangle^m$ for which in $N_{\boldsymbol{f}}$:

$$b(\sigma) g \boldsymbol{f}^{\boldsymbol{s}} = \sum_{k=1}^{r} P_k h_k \boldsymbol{f}^{\boldsymbol{s}}.$$

- For a polynomial $g \in K[\boldsymbol{x}]$ so that $g \otimes 1 \in M_{\boldsymbol{f}}$, $b_{\boldsymbol{f},g}$ is the monic minimal polynomial of σ on

$$\overline{M}_g := \frac{(V^0 D_Y)(g \otimes 1)}{(V^1 D_Y)(g \otimes 1)}.$$

- Since $(V^0 D_Y)\overline{g \otimes 1} \subseteq \overline{M}_{\boldsymbol{f}}^{(1)}$ is a quotient of $\overline{M}_g$, the polynomial $b_{\boldsymbol{f},g}$ is a multiple of $b_{\boldsymbol{f},g}^{(1)}$. When g is a unit, $b_{\boldsymbol{f},g} = b_{f,g}^{(1)}$ holds, which is not so in general.

Example

When $n = 3$ and $\boldsymbol{f} = \sum_{i=1}^{3} x_i^2$,

$$b_{\boldsymbol{f},x_1}(s) = (s+1)(s+\frac{5}{2}) \text{ and } b_{\boldsymbol{f},x_1}^{(1)}(s) = s+1.$$

In particular, $b_{\boldsymbol{f},x_1}^{(1)}$ strictly divides $b_{\boldsymbol{f},x_1}$.

Theorem (Shibuta)

For $g \in K[\boldsymbol{x}]$ and $c < m + \operatorname{lct}(\boldsymbol{f})$,
$g \in \mathcal{J}(\boldsymbol{f}^c) \;\Leftrightarrow\; c <$ every root of $b^{(m)}_{\boldsymbol{f},g}(-s)$.
In other words,

$$\mathcal{J}(\boldsymbol{f}^c) = \{g \in K[\boldsymbol{x}] : b^{(m)}_{\boldsymbol{f},g}(-c') = 0 \Rightarrow c < c'\}.$$

- Multivariate analogue of the ideal used to compute $\operatorname{Ann} f^s$:

$$I_{\boldsymbol{f}} = \langle t_i - f_i \mid 1 \le i \le r\rangle + \langle \partial_{x_j} + \textstyle\sum_{i=1}^r \frac{\partial f_i}{\partial t_j}\partial_{t_i} \mid 1 \le j \le n\rangle$$

- Let $I^*_{\boldsymbol{f}} \subset D_Y$ be the ideal of the $(-w,w)$-homogeneous elements of $I_{\boldsymbol{f}}$. Define:

$$J_{\boldsymbol{f}}(m) = \left(I^*_{\boldsymbol{f}} + D_Y \cdot \langle \boldsymbol{f}\rangle^m\right) \cap K[\boldsymbol{x},\sigma],$$

Lemma

$b^{(m)}_{\boldsymbol{f},g}$ is the monic polynomial $b(s) \in K[s]$ such that

$$\langle b(\sigma)\rangle = (J_{\boldsymbol{f}}(m) : g) \cap K[\sigma].$$

Algorithm with a “linear algebra” shortcut

Input: $\boldsymbol{f} = \{f_1, \ldots, f_r\} \subset K[\boldsymbol{x}]$, $c \in \mathbb{Q}$, $d_{\text{max}} \in \mathbb{Z}_{\geq 0}$.
Output: $\mathcal{J}(\boldsymbol{f}^c) \subset K[\boldsymbol{x}]$, if generated in degrees at most d_{max}.

1: $m \leftarrow \lceil \max\{c - \operatorname{lct}(\boldsymbol{f}), 1\} \rceil$.
2: $B \leftarrow$ a Gröbner basis of $J_{\boldsymbol{f}}(m)$ w.r.t. **any monomial order**.
3: Compute $b_{\boldsymbol{f},1}^{(m)} = \prod (s - c_i)^{\alpha(c_i)}$.
4: $b' \leftarrow \prod_{-c_i > c} (s - c_i)^{\alpha(c_i)}$.
5: Find a basis Q for the K-syzygies $(q_\alpha)_{|\alpha| \leq d_{max}}$ such that

$$\sum_{\alpha \in A} q_\alpha \operatorname{NF}_B(\boldsymbol{x}^\alpha b'(\sigma)) = 0.$$

6: **return** $\{\sum_{\alpha \in A} q_\alpha \boldsymbol{x}^\alpha : (q_\alpha) \in Q\}$.

Example

Consider $f = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)z$.
Saito: $\frac{5}{7}$ is a root of $b_f(-s)$ but not a jumping coefficient.
A long computation (about 10 hours) gives:

$$\mathcal{J}(f^c) = \begin{cases} \mathbb{C}[x, y] & \text{if } 0 \leq c < \frac{3}{7}, \\ \langle x, y, z \rangle & \text{if } \frac{3}{7} \leq c < \frac{4}{7}, \\ \langle x, y, z \rangle^2 & \text{if } \frac{4}{7} \leq c < \frac{2}{3}, \\ \langle z, x \rangle \cap \langle z, y \rangle \cap & \\ \langle y+z, x+z \rangle \cap \langle y+z, x-z \rangle \cap & \\ \langle y-z, x+z \rangle \cap \langle y-z, x-z \rangle & \text{if } \frac{2}{3} \leq c < \frac{6}{7}, \\ \langle z, x \rangle \cap \langle z, y \rangle \cap & \\ \langle y+z, x+z \rangle \cap \langle y+z, x-z \rangle \cap & \\ \langle y-z, x+z \rangle \cap \langle y-z, x-z \rangle \cap & \\ \langle z^3, yz^2, xz^2, xyz, y^3, x^3, x^2y^2 \rangle & \text{if } \frac{6}{7} \leq c < 1, \end{cases}$$

and $\mathcal{J}(f^c) = \langle \boldsymbol{f} \rangle \cdot \mathcal{J}(\boldsymbol{f}^{c-1})$ for all $c \geq 1$.

lct in the modern industrial society

Log-canonical threshold in statistics:

- S. Lin, "Asymptotic Approximation of Marginal Likelihood Integrals" (2010)
- S. Watanabe, "Algebraic Geometry and Statistical Learning Theory" (2009)

Main obstacle to application of a D-modules approach:

- Need to compute real lct.
- M. Saito, "On real log canonical thresholds" (2007):

$$\mathrm{rlct} \neq \mathrm{lct}$$