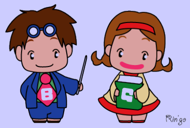
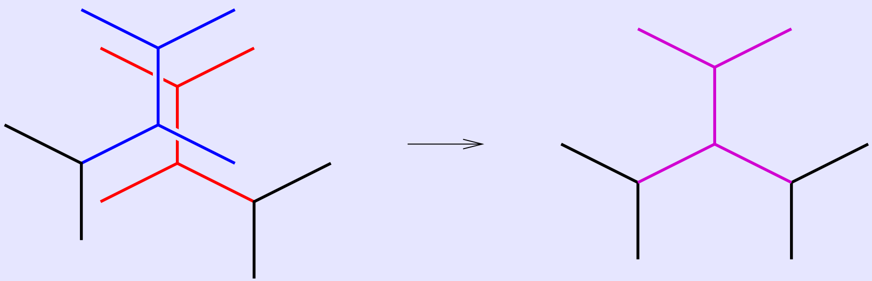


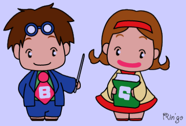
Harmony of Gröbner-Bases and Toric Fiber Products

Christian Haase

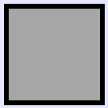
July 2, 2010

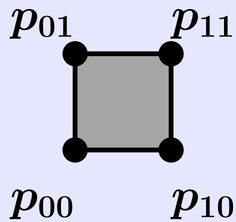


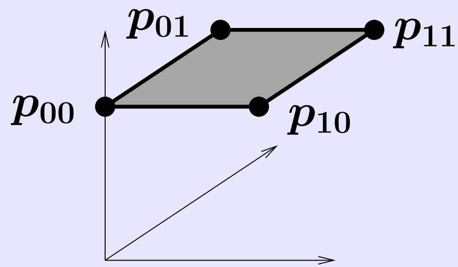
Joint with Kaie Kubjas & Andreas Paffenholz



1. Polytope ⇒ Variety

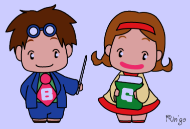






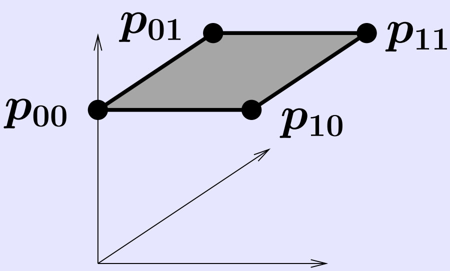
$P \subset \mathbb{R}^d$ $\quad$ $P \cap \mathbb{Z}^d$ $\quad$ $\mathcal{A} = (P \cap \mathbb{Z}^d) \times 1$

$$\begin{matrix} & & p_{00} & p_{10} & p_{01} & p_{11} \\ (r,s,t) & \longmapsto & (\;t, & rt, & st, & rst) \end{matrix}$$



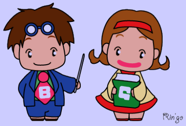
point configuration

$$\mathcal{A} = (P \cap \mathbb{Z}^d) \times 1$$



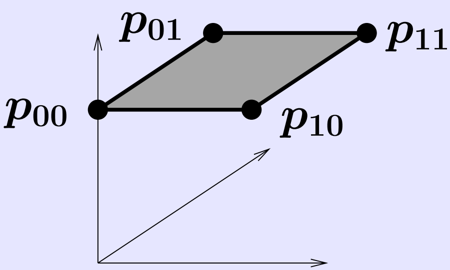
homomorphism

$$\begin{array}{ccc} \mathbb{C}[p_{ij}] & \longrightarrow & \mathbb{C}[r, s, t] \\ p_{00} & \longmapsto & t \\ p_{10} & \longmapsto & rt \\ p_{01} & \longmapsto & st \\ p_{11} & \longmapsto & rst \end{array}$$



point configuration

$$\mathcal{A} = (P \times 1) \cap \mathbb{Z}^d$$

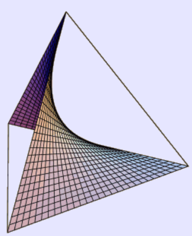


homomorphism

$$\begin{array}{ccc} \mathbb{C}[p_{ij}] & \longrightarrow & \mathbb{C}[r, s, t] \\ p_{00} & \mapsto & t \\ p_{10} & \mapsto & rt \\ p_{01} & \mapsto & st \\ p_{11} & \mapsto & rst \end{array}$$

linear relation

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

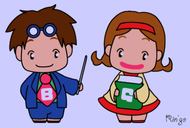


binomial relation

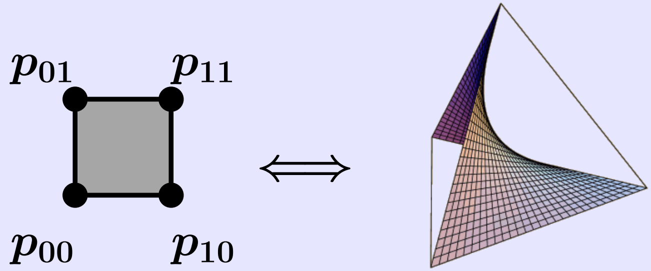
$$I_P = \langle p_{00}p_{11} - p_{01}p_{10} \rangle$$

toric variety

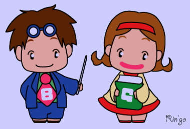
$$X_P \hookrightarrow \mathbb{P}^{4-1}$$



toric ideals



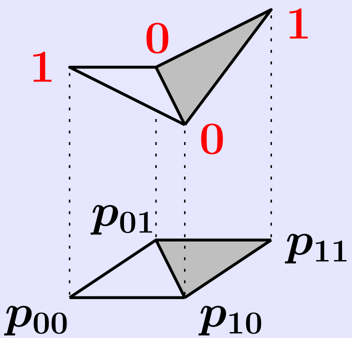
- semi-stabile reduction of families over curves [Kempf et al. 1973]
- g-Theorem for simplicial polytopes [Stanley 1980]
- McKay correspondence [Batyrev 1999]
- weak factorization of birational morphisms [Włodarczyk et al. 2003]
- log-linear statistical models

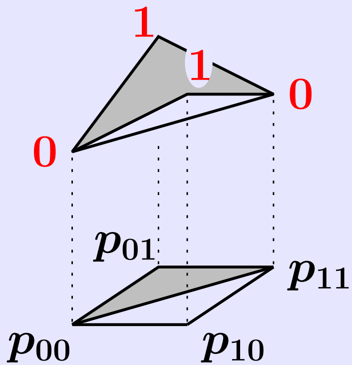


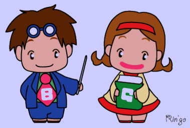
2. Toric Gröbner Bases

$I_P = \langle p_{00}p_{11} - p_{01}p_{10} \rangle$ has two Gröbner bases.

$\mathcal{G}_1 = \underline{p_{00}p_{11}} - p_{01}p_{10}$ $\qquad$ $\mathcal{G}_2 = p_{00}p_{11} - \underline{p_{01}p_{10}}$







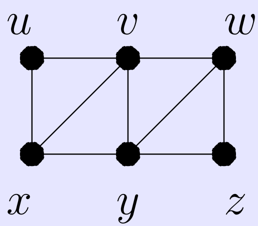
Theorem [Kapranov, Sturmfels, Zelevinski 1992]

I_P has a square-free initial ideal

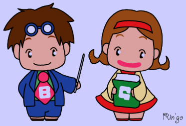
$$\Updownarrow$$

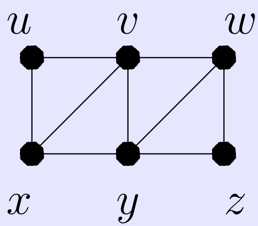
P has a regular unimodular triangulation

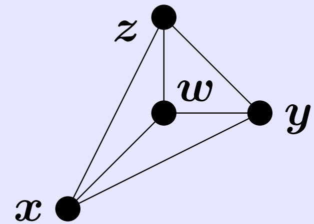
In that case, the Gröbner basis can be read off from the triangulation.



$$\underline{uw} - v^2\,,\ \underline{uy} - vx,$$
$$\underline{uz} - vy\,,\ \underline{vz} - wy,$$
$$\underline{wx} - vy,\ \underline{xz} - y^2$$

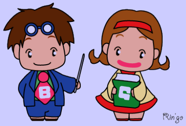


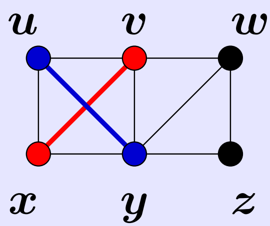


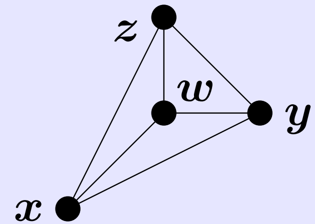


$$\underline{\boldsymbol{uw}} - \boldsymbol{v}^2,\ \underline{\boldsymbol{uy}} - \boldsymbol{vx},$$
$$\underline{\boldsymbol{uz}} - \boldsymbol{vy},\ \underline{\boldsymbol{vz}} - \boldsymbol{wy},$$
$$\underline{\boldsymbol{wx}} - \boldsymbol{vy},\ \underline{\boldsymbol{xz}} - \boldsymbol{y}^2$$

$$\underline{\boldsymbol{xyz}} - \boldsymbol{w}^3$$

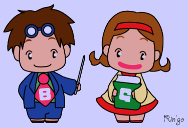


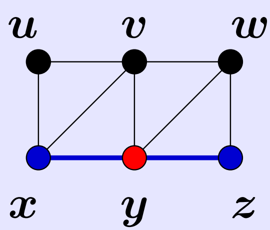


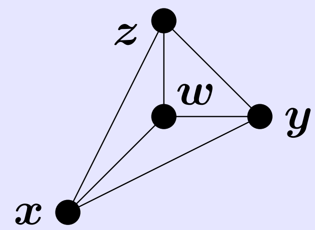


$$\underline{uw} - v^2\,,\ \underline{uy} - vx,$$
$$\underline{uz} - vy\,,\ \underline{vz} - wy,$$
$$\underline{wx} - vy,\ \underline{xz} - y^2$$

$$\underline{xyz} - w^3$$

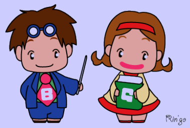


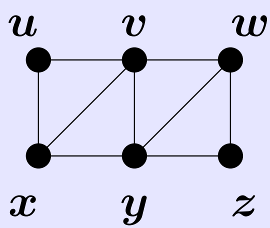


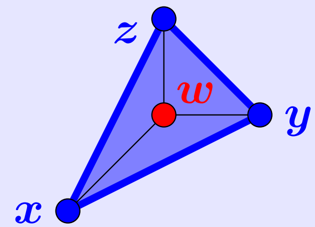


$$\underline{uw} - v^2,\ \underline{uy} - vx,$$
$$\underline{uz} - vy,\ \underline{vz} - wy,$$
$$\underline{wx} - vy,\ \underline{xz} - y^2$$

$$\underline{xyz} - w^3$$

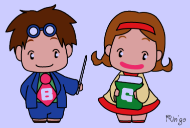






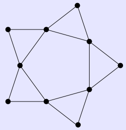
$$\underline{uw} - v^2,\ \underline{uy} - vx,$$
$$\underline{uz} - vy,\ \underline{vz} - wy,$$
$$\underline{wx} - vy,\ \underline{xz} - y^2$$

$$\underline{xyz} - w^3$$



Example [Ohsugi, Hibi 1999]

I_P without square-free initial ideal.

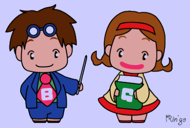


$\dim P = \dim X_P = 9, X_P \subset \mathbb{P}^{14}$

Proof:

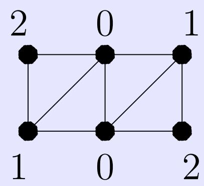
$$
\begin{aligned}
Gr_G = \{ & x_8x_{12}x_{15} - x_9x_{11}x_{13},\ x_6x_{11}x_{14} - x_7x_{12}x_{15},\ x_6x_8x_{14} - x_7x_9x_{13},\\
& x_4x_{13}x_{15} - x_5x_{11}x_{14},\ \underline{x_4x_9{x_{13}}^2 - x_5x_8x_{12}x_{14}},\ x_4x_8x_{12}{x_{15}}^2 - x_5x_9{x_{11}}^2x_{14},\\
& x_4x_6x_{13} - x_5x_7x_{12},\ x_4x_6x_8x_{15} - x_5x_7x_9x_{11},\ x_4{x_6}^2x_8x_{14} - x_5{x_7}^2x_9x_{12},\\
& x_2x_{12}x_{14} - x_3x_{13}x_{15},\ \underline{x_2x_9x_{11}x_{14} - x_3x_8{x_{15}}^2},\ x_2x_8{x_{12}}^2x_{14} - x_3x_9x_{11}{x_{13}}^2,\\
& \underline{x_2x_7{x_{12}}^2 - x_3x_6x_{11}x_{13}},\ x_2x_7x_9x_{12} - x_3x_6x_8x_{15},\ x_2x_7{x_9}^2x_{11}x_{13} - x_3x_6{x_8}^2{x_{15}}^2,\\
& x_2{x_7}^2x_9{x_{12}}^2 - x_3{x_6}^2x_8x_{11}x_{14},\ x_2x_6x_{11}{x_{14}}^2 - x_3x_7x_{13}{x_{15}}^2,\ x_2x_4x_{12} - x_3x_5x_{11},\\
& x_2x_4x_9x_{13} - x_3x_5x_8x_{15},\ x_2x_4x_6x_{14} - x_3x_5x_7x_{15},\ x_2{x_4}^2x_9{x_{13}}^2 - x_3{x_5}^2x_8x_{11}x_{14},\\
& x_2{x_4}^2x_6x_{13} - x_3{x_5}^2x_7x_{11},\ x_2{x_4}^2{x_6}^2x_8x_{14} - x_3{x_5}^2{x_7}^2x_9x_{11},\\
& {x_2}^2x_4x_9x_{12}x_{14} - {x_3}^2x_5x_8{x_{15}}^2,\ x_1x_{12}x_{14} - x_{10}x_{11}x_{13},\ x_1x_9x_{14} - x_8x_{10}x_{15},\\
& x_1x_7{x_{12}}^2x_{15} - x_6x_{10}{x_{11}}^2x_{13},\ x_1x_7x_9x_{12} - x_6x_8x_{10}x_{11},\ x_1x_7{x_9}^2x_{13} - x_6{x_8}^2x_{10}x_{15},\\
& \underline{x_1x_6{x_{14}}^2 - x_7x_{10}x_{13}x_{15}},\ x_1x_5x_{12}{x_{14}}^2 - x_4x_{10}{x_{13}}^2x_{15},\\
& x_1x_5{x_7}^2{x_9}^2x_{12} - x_4{x_6}^2{x_8}^2x_{10}x_{15},\ \underline{x_1x_4x_{12}x_{15} - x_5x_{10}{x_{11}}^2},\\
& x_1x_4x_9x_{13} - x_5x_8x_{10}x_{11}, x_1x_4{x_9}^2{x_{13}}^2 - x_5{x_8}^2x_{10}x_{12}x_{15}, x_1x_4x_6x_{14} - x_5x_7x_{10}x_{11},\\
& x_1x_4{x_6}^2{x_{14}}^2 - x_5{x_7}^2x_{10}x_{12}x_{15}, x_1{x_4}^2x_6x_{13}x_{15} - {x_5}^2x_7x_{10}{x_{11}}^2,\\
& x_1x_3x_{15} - x_2x_{10}x_{11}, x_1x_3x_9x_{13} - x_2x_8x_{10}x_{12},\\
& x_1x_3x_6x_{14} - x_2x_7x_{10}x_{12},\ x_1x_3x_5x_{14} - x_2x_4x_{10}x_{13},\\
& x_1x_3x_5x_7x_9 - x_2x_4x_6x_8x_{10},\ x_1{x_3}^2x_6x_{13}x_{15} - {x_2}^2x_7x_{10}{x_{12}}^2,\\
& x_1{x_3}^2x_5x_{15} - {x_2}^2x_4x_{10}x_{12},\ x_1{x_3}^2{x_5}^2x_7x_{15} - {x_2}^2{x_4}^2x_6x_{10}x_{13},\\
& {x_1}^2x_4x_9x_{12}x_{14} - x_5x_8{x_{10}}^2{x_{11}}^2,\ {x_1}^2x_3x_9x_{14} - x_2x_8{x_{10}}^2x_{11},\\
& {x_1}^2x_3x_7{x_9}^2x_{13} - x_2x_6{x_8}^2{x_{10}}^2x_{11},\ {x_1}^2x_3x_6{x_{14}}^2 - x_2x_7{x_{10}}^2x_{11}x_{13},\\
& {x_1}^2{x_3}^2x_5x_9x_{14} - {x_2}^2x_4x_8{x_{10}}^2x_{12}\}.
\end{aligned}
$$

□

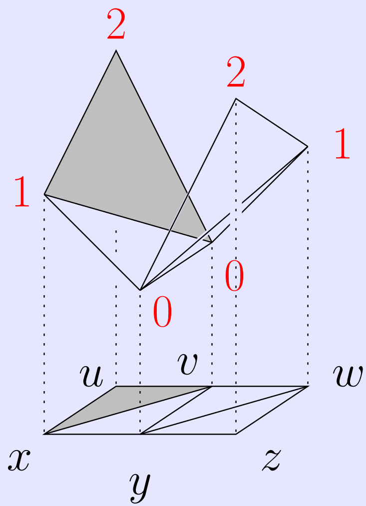


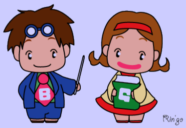
regular

weights at
lattice points/variables



regular triangulation





regular

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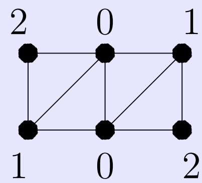
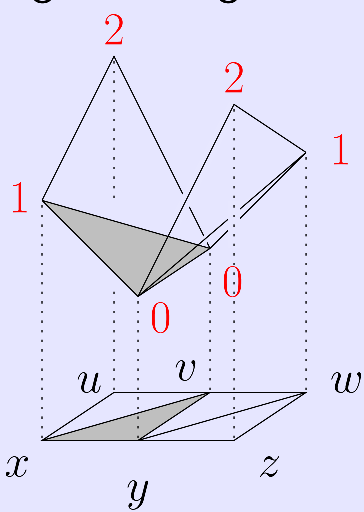
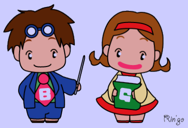


regular triangulation





regular

weights at
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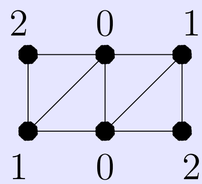
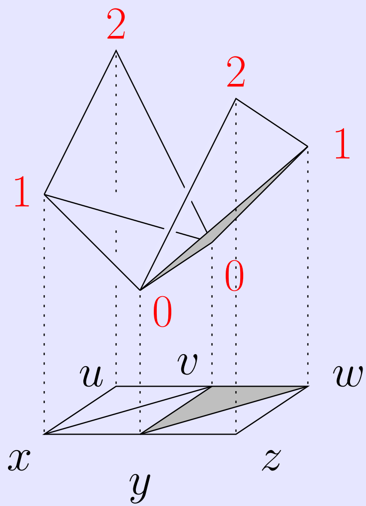
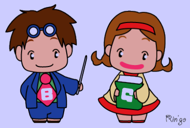


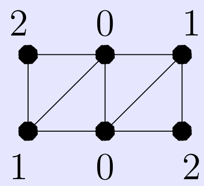
regular triangulation



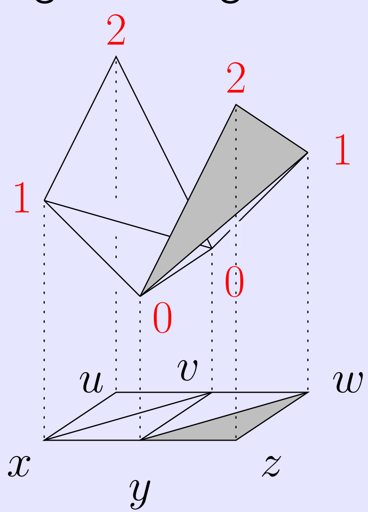


regular

weights at
lattice points/variables

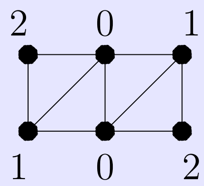


regular triangulation

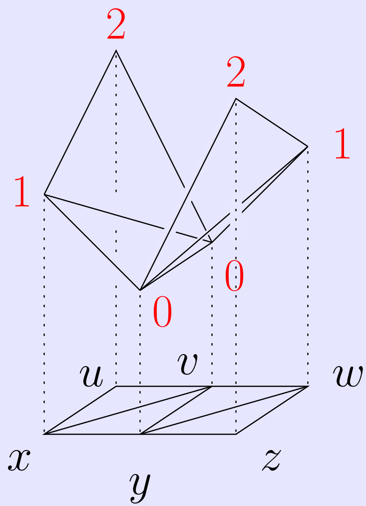


regular

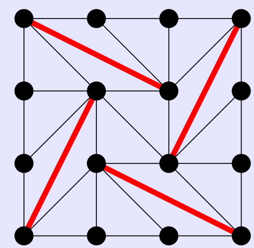
weights at
lattice points/variables

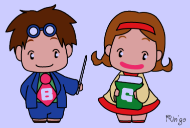


regular triangulation



not regular



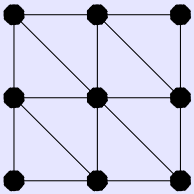


Definition

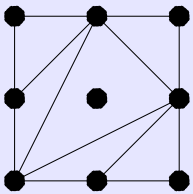
A lattice simplex $P \subset \mathbb{R}^d$ is *unimodular* if

$$\operatorname{vol} P = 1/d!\ .$$

A triangulation is *unimodular* if all its simplices are.

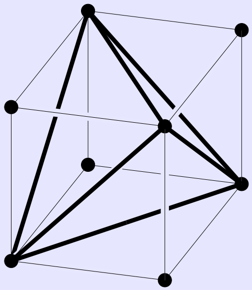


unimodular



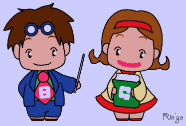
not unimodular

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$



not unimodular

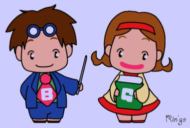
$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$



Positive Examples

- smooth surfaces [Bruns, Gubeladze, Trung '97]
- order polytopes [Santos '97, Ohsugi & Hibi '01]
- root systems [Ohsugi & Hibi '01]
- smooth, all lattice points vertices ['04]
- many smooth Fano varieties [, Piechnik, Paffenholz '04]
- Veronesoid embeddings
  [Stanley '77, Sturmfels '96, Lam, Postnikov '05]
- smooth 3×3 transportation polytopes
  [, Paffenholz '06]

3. Toric Fiber Products



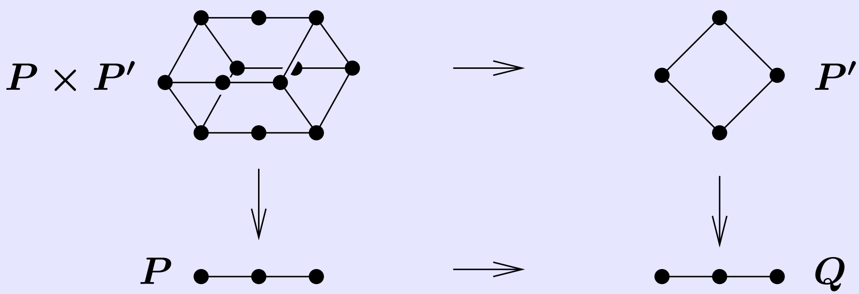
Definition [Sullivant 2007]

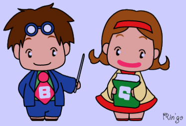
Suppose

$$P \xrightarrow{\pi} Q \xleftarrow{\pi'} P'$$

are lattice preserving polytope projections. Then the fiber product $P \times_Q P'$ is the polytope

$$\{(p, p') \in P \times P' \ : \ \pi(p) = \pi'(p')\} \ .$$





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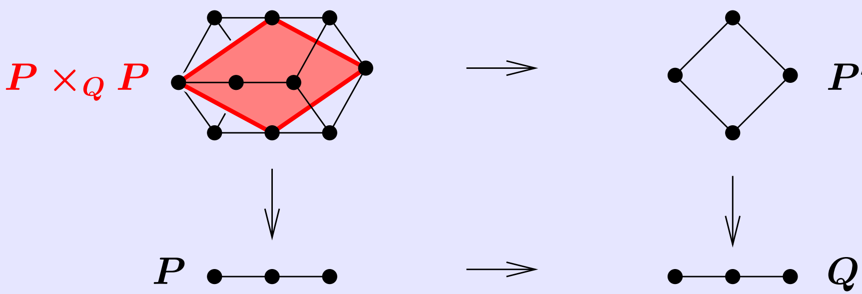
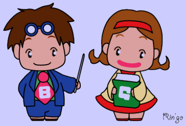
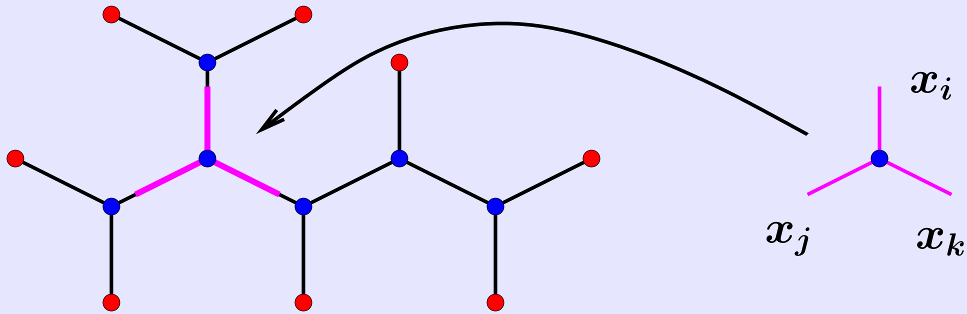




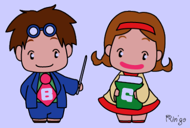
Example [Buczyńska, Wisniewski 2009]

Binary Jukes-Cantor models for trivalent trees (and other group based models).



$T = (V, E)$ trivalent tree with $V = L \cup N$.

$$P := \{x \in \{0,1\}^E \ : \ x_i \leq x_j + x_k,\ x_i + x_j + x_k \text{ even for all } T \hookleftarrow \text{(trivalent node)}\}$$



Example [Buczyńska, Wisniewski 2009]

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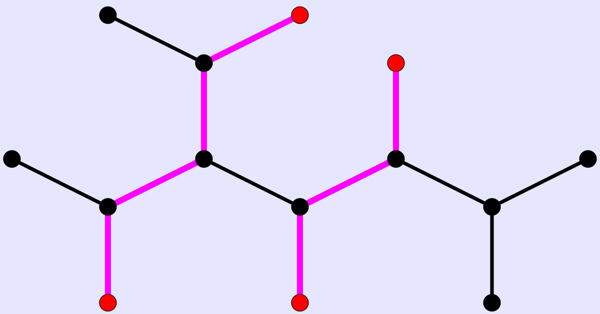
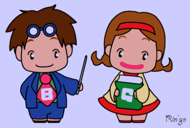


$T = (V, E)$ trivalent tree with $V = L \cup N$.

$$P := \operatorname{conv}\{\mathbb{1}_{E'} \ : \ E' \subset E \text{ joins even subset } L' \subseteq L\}$$



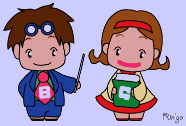
4. Gröbner-Bases for Toric Fiber Products

$$P \xrightarrow{\pi} Q \xleftarrow{\pi'} P' \text{ lattice preserving projections}$$

Theorem [, Kubjas, Paffenholz 2010?]

Q with regular unimodular triangulation $\mathcal{S}$,
P with regular unimodular triangulation refining $\pi^*\mathcal{S}$,
P' with regular unimodular triangulation refining $\pi'^*\mathcal{S}$

then $P \times_Q P'$ has a regular unimodular triangulation.

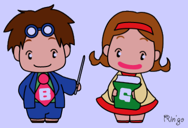


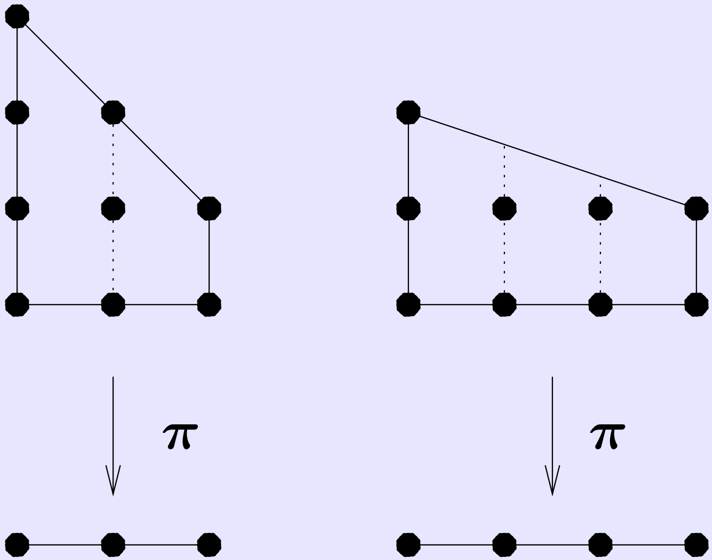
$$P \xrightarrow{\pi} \Delta \xleftarrow{\pi'} P' \text{ lattice preserving projections}$$

Corollary [Sullivant 2007]

P and P' with regular unimodular triangulations, then $P \times_\Delta P'$ has a regular unimodular triangulation.

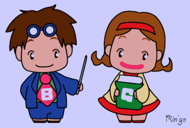
pull-back subdivisions $\pi^*\Delta$





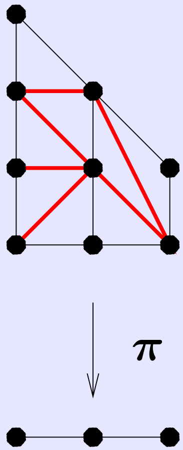
integral

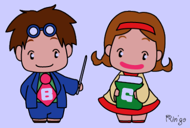
not integral



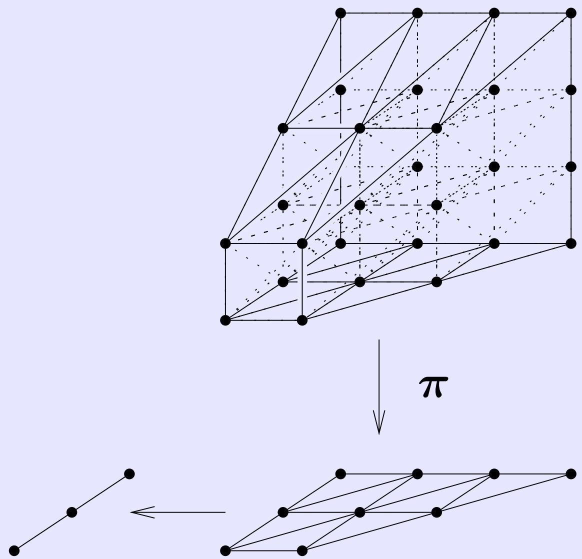
Chimney-Lemma [Dais, , Ziegler 2001]
Q with unimodular triangulation $\mathcal{T}$,
$\dim P = \dim Q + 1$,

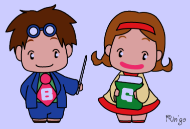
then every full refinement of $\pi^*\mathcal{T}$ yields a unimodular triangulation.



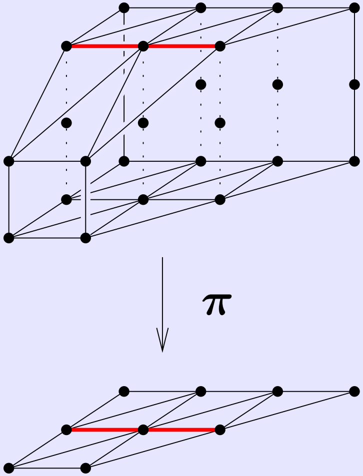


easy

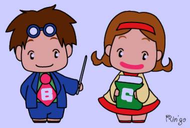




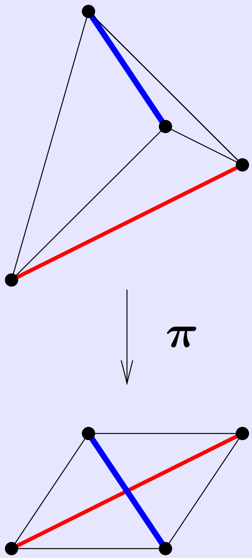
not so easy

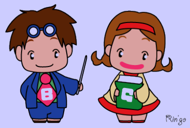


(push-forward subdivision $\pi_*\Delta$)



impossible





products

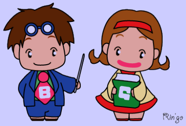
P, P' with regular
unimodular triangulation,

$\searrow$

subdivision of $P \times P'$ into
products of unimodular simplices

$\searrow$

unimodular
triangulation
of $P \times P'$

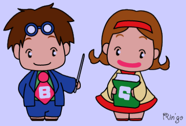


Lemma

$\Delta_d \xrightarrow{\pi} \Delta_{d''} \xleftarrow{\pi'} \Delta_{d'}$ lattice preserving projections,

then $\Delta_d \times_{\Delta_{d''}} \Delta_{d'}$ is integral and compressed.

□



5. Degree Bounds

. . . just kidding