Toric ideals of small matroids are generated in degree 2

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Old and New Results

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The toric ideal of any matroid on a ground set of size ≤ 9 is generated by quadrics.

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Theorem (Kashiwabara 10)

The toric ideal of a matroid of rank 3 is generated by quadrics.

Matroids

Definition

A matroid M is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ is a collection of subsets of E satisfying

- B1 $\mathcal{B} \neq \emptyset$ and no member of $\mathcal{B}$ is a subset of another,
- B2 If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$, then there exists $f \in B_2$ such that $B_1 - e + f \in \mathcal{B}$.

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B2 is the basis exchange axiom.

Matroids

Example

$M = (E, \mathcal{B})$, $E = \{1, 2, 3, 4, 5, 6\}$,
$\mathcal{B} = \{12, 14, 24, 25, 26, 45, 46\}$ (ij denotes $\{i, j\}$)

For $12, 46 \in \mathcal{B}$, $12 - 2 + 4 = 14 \in \mathcal{B}$.

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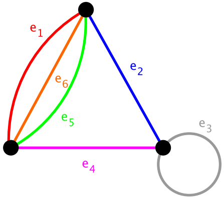


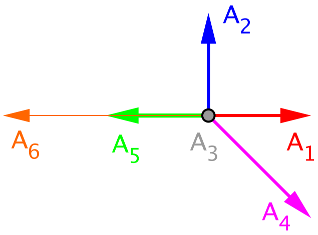
M is a graphic matroid.

Matroids

Example

$$E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right\}$$

$\mathcal{B}$ = { all bases of this vector configuration }



Double Swaps

Proposition

Let $M = (E, \mathcal{B})$ *be a matroid and* $B_1, B_2 \in \mathcal{B}$*. For every* $e \in B_1$ *there exists* $f \in B_2$ *such that* $D_1 = B_1 - e + f$ *and* $D_2 = B_2 - f + e$ *are bases.*

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Example

$E = \{\,1,\ 2,\ 3,\ 4,\ 5,\ 6\,\}, \quad \mathcal{B} = \{\,12,\ 14,\ 24,\ 25,\ 26,\ 45,\ 46\,\}$

$\{14, 25\} \leftrightarrow \{12, 45\}$ is a double swap, but $\{14, 25\} \leftrightarrow \{15, 24\}$ is not a double swap.

White's Conjecture

Conjecture (Neil White, 1977)

Let $M = (E, \mathcal{B})$ be a matroid. For every $m \geq 2$, any two collections of bases $\{B_1, B_2, \ldots, B_m\}$ and $\{D_1, D_2, \ldots, D_m\}$ such that $\biguplus B_i = \biguplus D_i$ can be connected by double swaps.

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- $I_M = \langle x_{B_{i_1}} x_{B_{i_2}} \cdots x_{B_{i_k}} - x_{B_{j_1}} x_{B_{j_2}} \cdots x_{B_{j_k}} :\ \dot\bigcup_{s=1}^{k} B_{i_s} = \dot\bigcup_{s=1}^{k} B_{j_s} \rangle$.

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Conjecture (White's conjecture translated)

The toric ideal I_M is generated by $x_B x_{B'} - x_D x_{D'}$ where $\{B, B'\} \leftrightarrow \{D, D'\}$ is a double swap.

Variants of White's conjecture

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The toric ideal I_M has a Gröbner basis consisting of quadratic binomials.

Why?

- Quadratic generators $\longrightarrow$ few generators
- Quadratic Gröbner basis consisting of double swaps $\longrightarrow$ unimodular triangulation of matroid polytopes (still open; see David Haws' work)
- It fits nicely to other similar questions.
- The closure of torus orbit of a generic point $K \in k^{r \times n}$ in the Grassmannian $\mathbb{G}(r, n)$ is the toric variety corresponding to the matroid of bases of K.

What is known

Theorem (Sturmfels 96)

If M is a uniform matroid then the double swaps form a Gröbner basis of I_M.

Theorem (Blasiak 08)

If M is a graphical matroid then I_M is generated by double swaps.

Theorem (Ohsugi-Hibi 00, Blum 01, HY 10)

If $\operatorname{rank}(M) = 2$ then I_M is generated by double swaps (actually form a Gröbner basis).

Theorem (Kashiwabara 10)

If $\operatorname{rank}(M) = 3$ then I_M is generated by double swaps.

size ≤ 9

Mayhew and Royle (2008) have classified all non-isomorphic matroids on ≤ 9 elements.

r/n	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2			1	3	7	13	23	37	58	87
3				1	4	13	38	108	325	1275
4					1	5	23	108	940	190214
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	1	2	4	8	17	38	98	306	1724	383172

Proposition

$I_{M^*} = I_M$ where M^* is the dual matroid of M.

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There are at least 2.5×10^{12} matroids of rank 10.

Blasiak's Reduction

Proposition (Blasiak 08)

Let b be binomial of degree $m \geq 3$ in I_M of a matroid M of rank r

$$b = x_{B_{i_1}} x_{B_{i_2}} \cdots x_{B_{i_m}} - x_{B_{j_1}} x_{B_{j_2}} \cdots x_{B_{j_m}}.$$

Then we get a new matroid M' of rank r on rm elements and a binomial

$$b' = x_{D_{i_1}} x_{D_{i_2}} \cdots x_{D_{i_m}} - x_{D_{j_1}} x_{D_{j_2}} \cdots x_{D_{j_m}}$$

where both $\{D_{i_1}, D_{i_2}, \ldots, D_{i_m}\}$ and $\{D_{j_1}, D_{j_2}, \cdots, D_{j_m}\}$ are partitions of the ground set of M'. Moreover, b is connected by double swaps of M if and only if b' is connected by double swaps of M'.

rank = 2

Follows from an easy argument using Blasiak's reduction. We illustrate for $m = 4$:

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We need to connect 12 34 56 78 to 13 25 47 68.

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We need to connect $12\,34\,56\,78$ to $13\,25\,47\,68$.

Two double swaps: $(56, 78) \leftrightarrow (57, 68)$ or $(56, 78) \leftrightarrow (67, 58)$.

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Two double swaps: $(56, 78) \leftrightarrow (57, 68)$ or $(56, 78) \leftrightarrow (67, 58)$.

Two other double swaps: $(47, 68) \leftrightarrow (46, 78)$ or $(47, 68) \leftrightarrow (67, 48)$.

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The induction step uses a non-trivial matroid result which is a consequence of Matroid Partition Theorem.

THANK YOU