Non-very ample configurations arising from contingency tables

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Outline

Toric ideals

$\mathbb{N}$: the set of nonnegative integers

$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$: configuration
(Assume that $\exists \mathbf{w} \in \mathbb{R}^d$ s.t. $\mathbf{w} \cdot \mathbf{a}_i = 1$ for $\forall i$)

$K[\mathbf{t}] := K[t_1, t_2, \ldots, t_d]$: polynomial ring over a field K

$(\mathbf{a} = (2,4,0,1) \in \mathbb{N}^4 \Rightarrow \mathbf{t}^{\mathbf{a}} := t_1^2 t_2^4 t_4 \in K[t_1, t_2, t_3, t_4])$

$K[A] := K[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ $(\subset K[\mathbf{t}])$: semigroup ring

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$$I_A = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in K[\mathbf{x}] \mid A\mathbf{u} = A\mathbf{v} \rangle \quad \text{toric ideal of } A$$

Here, we regard $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ as a $d \times n$ matrix.

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Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \subset \mathbb{N}^5$$

$$K[A] = K[t_1t_3, t_1t_4, t_1t_5, t_2t_3, t_2t_4, t_2t_5]$$

$$I_A = \langle x_1x_5 - x_2x_4,\ x_1x_6 - x_3x_4,\ x_2x_6 - x_3x_5 \rangle$$

I_A is generated by 2-minors of $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$.

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Toric ideals and Statistics

Theorem (Diaconis–Sturmfels)

"a set of binomial generators of I_A" = "a Markov basis"

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = \begin{pmatrix} u_{11}+u_{12}+u_{13} \\ u_{21}+u_{22}+u_{23} \\ u_{11}+u_{21} \\ u_{12}+u_{22} \\ u_{13}+u_{23} \end{pmatrix}$$

$$x_1x_5 - x_2x_4 \in I_A \longleftrightarrow (1,-1,0,-1,1,0)^T \longleftrightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

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Normal configurations and very ample configurations

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) holds for the following:
(However, none of $\Leftarrow$ is false in general.)

(i) A is *unimodular*, i.e., the initial ideal of I_A is generated by squarefree monomials with respect to any monomial order

(ii) A is *compressed*, i.e., the initial ideal of I_A is generated by squarefree monomials with respect to any reverse lexicographic order

(iii) There exists a monomial order $<$ such that the initial ideal of I_A with respect to $<$ is generated by squarefree monomials

(iv) $K[A]$ is *normal*, i.e., $\mathbb{N}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A$

(v) $K[A]$ is *very ample*, i.e., $\sharp|(\mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A) \setminus \mathbb{N}A| < \infty$.

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Combinatorial pure subrings

Definition

P_A: the convex hull of A

$B \subset A$: subset

$K[B]$ is combinatorial pure subring of $K[A]$

$\iff \exists$ face F of P_A s.t. $B = A \cap F$

For example, $K[B]$ is a combinatorial pure subring of $K[A]$ if

$$K[B] = K[A] \cap K[t_{i_1}, \ldots, t_{i_s}]$$

(This is the original definition by Ohsugi–Herzog–Hibi.)

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Fundamental binomials

Definition

$f \in I_A$: binomial

f is fundamental

$\iff \exists$ combinatorial pure subring $K[B]$ of $K[A]$ s.t $I_B = \langle f \rangle$

Lemma

If $g = u - v \in K[\mathbf{x}]$ is a binomial such that neither u nor v is squarefree and if $I_A = \langle g \rangle$, then $K[A]$ is not very ample.

Corollary

If I_A possesses a fundamental binomial $g = u - v$ such that neither u nor v is squarefree, then $K[A]$ is not very ample.

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Remark

The converse of Corollary is not true.

Example

$K[A] = K[t_1t_2, t_1t_3, t_2t_3, t_2t_4, t_3t_4, t_4t_5, t_4t_6, t_5t_6, t_5t_7, t_6t_7]$

Then $K[A]$ is not very ample.

However I_A is generated by the set of fundamental binomials

$$\{x_1x_5 - x_2x_4, x_6x_{10} - x_7x_9, x_3x_6x_7 - x_4x_5x_8\}.$$

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Lawrence lifting

$$\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ I_n & I_n \end{pmatrix}$$: Lawrence lifting of A

Corollary

Let $K[A]$ be a semigroup ring and let $K[\Lambda(A)]$ its Lawrence lifting. Then, the following conditions are equivalent:

1. $K[A]$ *is unimodular;*
2. $K[\Lambda(A)]$ *is unimodular;*
3. $K[\Lambda(A)]$ *is normal;*
4. $K[\Lambda(A)]$ *is very ample.*

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Contingency tables

$r_1 \geq r_2 \geq \cdots \geq r_n \geq 2$

We study configurations arising from no n-way interaction models for $r_1 \times r_2 \times \cdots \times r_n$ contingency tables.

$$A_{r_1 r_2 \cdots r_n} = \left\{ \mathbf{e}^{(1)}_{i_2 i_3 \cdots i_n} \oplus \mathbf{e}^{(2)}_{i_1 i_3 \cdots i_n} \oplus \cdots \oplus \mathbf{e}^{(n)}_{i_1 i_2 \cdots i_{n-1}} \;\middle|\; i_k \in \{1, 2, \ldots, r_k\}, 1 \leq k \leq n \right\}$$

Here $\mathbf{e}^{(k)}_{j_1 j_2 \cdots j_{n-1}}$ is a unit vector of $\mathbb{Z}^{r_1 \times \cdots r_{k-1} \times r_{k+1} \cdots \times r_n}$.

Example

$$A_{23} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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Contingency tables

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Example

$$A_{23} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Contingency tables

$r_1 \geq r_2 \geq \cdots \geq r_n \geq 2$

We study configurations arising from no n-way interaction models for $r_1 \times r_2 \times \cdots \times r_n$ contingency tables.

$A_{r_1 r_2 \cdots r_n} =$

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Example

$$A_{23} = \left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

Contingency tables

A_{222}	○ ○ ○	○ ○ ×	○ × ○	○ × ×	× ○ ○	× ○ ×	× × ○	× × ×
○○	1	1						
○×			1	1				
×○					1	1		
××							1	1
○○	1				1			
○×		1				1		
×○			1				1	
××				1				1
○○	1		1					
○×		1		1				
×○					1		1	
××						1		1

Unimodular & compressed

Fact

$A_{r_1 \cdots r_n 2}$ *is the Lawrence lifting of* $A_{r_1 \cdots r_n}$.

It is known that $A_{r_1 r_2}$ is unimodular

$\Rightarrow A_{r_1 r_2 2}$ is unimodular

$\Rightarrow A_{r_1 r_2 22}$ is unimodular

$\vdots$

$\Rightarrow A_{r_1 r_2 2 \cdots 2}$ is unimodular

Theorem (Sullivant)

$A_{r_1 r_2 \cdots r_n}$ *is compressed* $\Leftrightarrow$ *one of the following holds:*

- $n = 2$ *(unimodular)*
- $n \geq 3$ *and* $r_3 = 2$ *(unimodular)*
- $n = 3$ *and* $r_1 \geq r_2 = r_3 = 3$ *(not unimodular)*

Unimodular & compressed

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$\vdots$
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$A_{r_1 r_2 \cdots r_n}$ is compressed $\Leftrightarrow$ one of the following holds:

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- $n = 3$ and $r_1 \geq r_2 = r_3 = 3$ (not unimodular)

Unimodular & compressed

Fact

$A_{r_1 \dots r_n 2}$ is the Lawrence lifting of $A_{r_1 \dots r_n}$.

It is known that $A_{r_1 r_2}$ is unimodular
$\Rightarrow A_{r_1 r_2 2}$ is unimodular
$\Rightarrow A_{r_1 r_2 22}$ is unimodular

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Classification 1

Classification

$r_1 \times r_2$ $r_1 \times r_2 \times 2 \times \cdots \times 2$	*unimodular*
$r_1 \times 3 \times 3$	*compressed, not unimodular*
$5 \times 5 \times 3$ $5 \times 4 \times 3$ $4 \times 4 \times 3$	*not compressed*
otherwise, i.e., $n \geq 4$ *and* $r_3 \geq 3$ $n = 3$ *and* $r_3 \geq 4$ $n = 3, r_3 = 3, r_1 \geq 6$ *and* $r_2 \geq 4$	*not compressed*

Very ample

Lemma

$s_i \le r_i$ for all $1 \le i \le n$
$\Longrightarrow K[A_{s_1\cdots s_n}]$ is a combinatorial pure subring of $K[A_{r_1\cdots r_n}]$.

It is known that A_{333} is not unimodular

$\Downarrow$

A_{3332} is not very ample $\Rightarrow K[A_{r_1 r_2 r_3 r_4}]$ is not very ample $(r_3 \ge 3)$

$\Downarrow$

A_{33322} is not very ample $\Rightarrow K[A_{r_1\cdots r_5}]$ is not very ample $(r_3 \ge 3)$

$\vdots$

$A_{3332\cdots 2}$ is not very ample $\Rightarrow K[A_{r_1\cdots r_n}]$ is not very ample $(r_3 \ge 3)$

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$A_{3332 \cdots 2}$ is not very ample $\Rightarrow K[A_{r_1 \cdots r_n}]$ is not very ample ($r_3 \geq 3$)

Very ample

Lemma

For $I_{A_{643}}$ and $I_{A_{444}}$, there exists a fundamental binomial all of whose monomials are not squarefree.

By Corollary, $K[A_{643}]$ is not very ample
$\Longrightarrow K[A_{r_1 r_2 3}]$ ($r_1 \geq 6$ and $r_2 \geq 4$) is not very ample

By Corollary, $K[A_{444}]$ is not very ample
$\Longrightarrow K[A_{r_1 r_2 r_3}]$ ($n = 3$ and $r_1 \geq r_2 \geq r_3 \geq 4$) is not very ample

Very ample

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Classification 2

Classification

$r_1 \times r_2$ $r_1 \times r_2 \times 2 \times \cdots \times 2$	*unimodular*
$r_1 \times 3 \times 3$	*compressed, not unimodular*
$5 \times 5 \times 3$ $5 \times 4 \times 3$ $4 \times 4 \times 3$	*not compressed*
otherwise, i.e., $n \geq 4$ *and* $r_3 \geq 3$ $n = 3$ *and* $r_3 \geq 4$ $n = 3$, $r_3 = 3$, $r_1 \geq 6$ *and* $r_2 \geq 4$	*not normal, not very ample*

Normal

The computational proofs that

$$5 \times 5 \times 3 \text{ contingency tables is normal}$$

are given in the paper

W. Bruns, R. Hemmecke, B. Ichim, M. Köppe, C. Söger, Challenging computations of Hilbert bases of cones associated with algebraic statistics, Jan. 2010.

The computations are based on extensions of the packages LattE-4ti2 and Normaliz.

Classifications 3

Classification

$r_1 \times r_2$ $r_1 \times r_2 \times 2 \times \cdots \times 2$	*unimodular*
$r_1 \times 3 \times 3$	*compressed,* *not unimodular*
$5 \times 5 \times 3$ $5 \times 4 \times 3$ $4 \times 4 \times 3$	*normal* *(by* `4ti2 & Normaliz`*)* *not compressed*
otherwise, i.e., $n \geq 4$ *and* $r_3 \geq 3$ $n = 3$ *and* $r_3 \geq 4$ $n = 3$, $r_3 = 3$, $r_1 \geq 6$ *and* $r_2 \geq 4$	*not normal,* *not very ample*