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Hermite polynomial aliasing in Gaussian quadrature

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Abstract and a reference

Computational methods based on polynomial algebra software have been used in Statistics for Design of Experiments (DoE) and for other problems in statistical modeling.

In this approach to DoE, the set of design points is described as the solution of a system of polynomial equations and the identification of various classes of models is computed by the use of special bases of the polynomial ideal generated.

Here we present the first results of a research in progress in which we explore the applicability of these ideas when the defining equations are derived from Hermite polynomials.

A recent overview of this new field, termed Algebraic Statistics, and the first mention of the application to polynomial chaos are in

- P. Gibilisco, E. Riccomagno, M. Rogantin, H.P. Wynn, eds., *Algebraic and Geometric Methods in Statistics*, Cambridge University Press, 2010.

A representation of some polynomials, including those of degree $2n-1$, as sum of an element in the polynomial ideal generated by the roots of the Hermite polynomial of degree n and of a reminder, suggests a folding of multivariate polynomials over a finite set of points.

From this, the expectation of some polynomial combinations of random variables normally distributed is computed.

This is related to quadrature formulas and has strong links with designs of experiments. Hermite polynomials are key tools in many areas: chaos expansions, Malliavin calculus, SODE and SPDE, p-rough paths, ...

I. Hermite polynomials and Stein-Markov operator

Definition

1. Define $\delta f(x) = xf(x) - f'(x) = -e^{x^2/2}\frac{d}{dx}\left(f(x)e^{-x^2/2}\right)$. If $Z \sim \mathcal{N}(0,1)$,

$$\mathrm{E}\left(g(Z)\delta^n f(Z)\right) = \mathrm{E}\left(d^n g(Z) f(Z)\right),$$

   i.e. δ is the transpose of the derivative w.r.t. the standard Gaussian measure.

2. Define $H_0 = 1$, $H_n(x) = \delta^n 1$, $n > 0$, e.g.

$$H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x, H_4(x) = x^4 - 6x^2 + 3, \ldots$$

Properties

1. The transposition formula shows that the H_n's are orthogonal
2. $d\delta - \delta d = \mathrm{id}$, $dH_n = nH_{n-1}$, $\mathrm{E}(H_n^2(Z)) = n!$, $H_{n+1} = xH_n - nH_{n-1}$.

- P. Malliavin, *Integration and probability*, Springer-Verlag, New York, 1995, with the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, foreword by Mark Pinsky.

Theorem (Zeros of H_n)

1. *The Hermite polynomial H_n, $n \geq 1$, has n distinct real roots*
2. *which are separated by those of H_{n+1}.*

Theorem (Linear structure)

$\{H_0, \dots, H_n\}$ *spans the set of all polynomials of degree at most n.*

Theorem (Ring structure)

1. $H_k H_n = H_{n+k} + \sum_{i=1}^{n \wedge k} \binom{n}{i}\binom{k}{i} i! H_{n+k-2i}$, $n, k \geq 1$.
2. *If* $H_n(x) = 0$, *then* $H_{n+k}(x) + \sum_{i=1}^{n \wedge k} \binom{n}{i}\binom{k}{i} i! H_{n+k-2i}(x) = 0$, $n \geq 1$.

In statistical language, item 3 shows an aliasing relation on the design

$$\mathcal{D}_n = \{x \colon H_n(x) = 0\}$$

- W. Gautschi, *Orthogonal polynomials: computation and approximation*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2004.

Product formula

Let $\langle \phi, \psi \rangle = \mathrm{E}(\phi(Z)\psi(Z))$ and $h < k$. Then

$$\langle H_k H_h, \psi \rangle = \langle H_h, H_k \psi \rangle = \langle 1, d^h(H_k \psi) \rangle = \sum_{i=0}^{h} \langle 1, \binom{h}{i} d^i H_k d^{h-i} \psi \rangle$$

$$= \langle 1, H_k d^h \psi \rangle + \sum_{i=1}^{h} \langle 1, \binom{h}{i} d^i H_k d^{h-i} \psi \rangle$$

$$= \langle H_{h+k}, \psi \rangle + \sum_{i=1}^{h} \binom{h}{i} k(k-1)\dots(k-i+1) \langle H_{h+k-2i}, \psi \rangle$$

Example: $H_2 H_1 = (x^2 - 1)x = H_3 + 2H_1$ and in particular

$$H_k^2 = H_{2k} + \sum_{i=1}^{k} \binom{k}{i} k(k-1)\dots(k-i+1) H_{2k-2i}$$

$$\mathrm{E}(H_k^2(Z)) = \binom{k}{1} k(k-1)\dots 1 = k!$$

$$\mathrm{E}(H_k(Z) H_h(Z)) = 0$$

Algebraic DoE: basics

- Given a finite set $\mathcal{D}$ of distinct points in $\mathbb{R}^d$ we consider the design ideal

$$\begin{aligned}&\langle f \in \mathbb{R}[x_1, \dots, x_d] : f(d) = 0 \text{ for all } d \in \mathcal{D}\rangle \\ =&\langle f_1, \dots, f_p \in \mathbb{R}[x_1, \dots, x_d]\rangle\end{aligned}$$

- Two polynomials h, k, are aliased if $h - k$ is zero on $\mathcal{D}$, i.e. if $h - k$ belong to the design ideal.
- A fraction is a subset $\mathcal{F}$ of $\mathcal{D}$. Its design ideal is obtained by adding new equations $g_1, \dots, g_l$, called defining equations.
- The indicator polynomial of the fraction $\mathcal{F}$ in $\mathcal{D}$ is a polynomial whose restriction to $\mathcal{D}$ is the indicator function of the fraction.
- This is made operative by notions from Algebraic Geometry such as term-order, Gröbner basis, normal form, ... and algebraic software such as CoCoA, Maple, Singular, 4ti2, Matematica, Maxima, Macaulay2, ...

- G. Pistone, E. Riccomagno, H.P. Wynn, *Algebraic statistics. Computational commutative algebra in statistics*, Monographs on Statistics and Applied Probability, Chapman & Hall/CRC, Boca Raton, FL, 2001.
- D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*, 2nd edn., Springer-Verlag, New York, 1997.

- Let f be a polynomial in one variable with real coefficients and by polynomial division $f(x) = q(x)H_n(x) + r(x)$ where r has degree smaller than H_n and $r(x) = f(x)$ on $H_n(x) = 0$. The $n-1$ degree polynomial r is the remainder or normal form $\mathrm{NF}(f) = r$.
- Let $f \in \mathbb{R}[x_1, \ldots, x_d]$, τ a term-ordering, $f_1, \ldots, f_p \in \mathbb{R}[x_1, \ldots, x_d]$ a τ-Gröbner basis. Then there exists $q_1, \ldots, q_p$ and a unique r such that

$$f(x) = \sum_{i=1}^{p} q_i(x) f_i(x) + r(x)$$

  and (the learding term of) $r(x)$ is smaller than (the leading term of) f_i $(i = 1, \ldots p)$.

Aliasing computation

- The computation of the normal form introduces a notion of confounding. For example from $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ we obtain $H_{n+1}(x) \equiv -nH_{n-1}(x)$ where $\equiv$ stands for equality over $\mathcal{D}_n = \{x : H_n(x) = 0\}$, that is remainder of division by H_n.
- In general let $H_{n+k} \equiv \sum_{j=0}^{n-1} h_j^{n+k} H_j$ be the representation of H_{n+k} at $\mathcal{D}_n$. Substitution in the product formula gives

$$\begin{aligned} \mathrm{NF}(H_{n+k}) &\equiv -\sum_{i=1}^{n\wedge k} \binom{n}{i}\binom{k}{i} i!\, \mathrm{NF}(H_{n+k-2i}) \\ &= -\sum_{i=1}^{n\wedge k} \binom{n}{i}\binom{k}{i} i! \sum_{j=0}^{n-1} h_j^{n+k-2i} H_j \end{aligned}$$

Equating coefficients gives a general recursive formula

$$h_j^{n+k} = -\sum_{i=1}^{n\wedge k} \binom{n}{i}\binom{k}{i} i!\, h_j^{n+k-2i}$$

The first confounding relationships are

k	expansion
1	$-nH_{n-1}$
2	$-n(n-1)H_{n-2}$
3	$-n(n-1)(n-2)H_{n-3} + 3nH_{n-1}$
4	$-n(n-1)(n-2)(n-3)H_{n-4} + 8n(n-1)H_{n-2}$
5	$-\frac{n!}{(n-5)!}H_{n-5} + 5nH_{n-1} + 15n(n-1)(n-2)H_{n-3}$
6	$-\frac{n!}{(n-6)!}H_{n-6} + 24n(n-1)(n-2)(n-3)H_{n-4} + 10n(n-1)(2n-5)H_{n-2}$

- For $f = \sum_{i=0}^{n+1} c_i(f)H_i$, we have $k = 1$ and

$$
\begin{aligned}
\mathrm{NF}(f) &= \sum_{i=0}^{n-1} c_i(f)H_i + \underline{c_n(f)H_n} + c_{n+1}(f)\,\mathrm{NF}(H_{n+1}) \\
&\equiv \sum_{i=0}^{n-2} c_i(f)H_i + \left(c_{n-1}(f) - nc_{n+1}(f)\right) H_{n-1}
\end{aligned}
$$

II. Expectation and NF

- Let f be a polynomial in one variable with real coefficients and by polynomial division $f(x) = q(x)H_n(x) + r(x)$ where r has degree smaller than H_n and $r(x) = f(x)$ on $H_n(x) = 0$. The $n-1$ degree polynomial r is the remainder or normal form $\mathrm{NF}(f) = r$.
- Then for $Z \sim \mathcal{N}(0,1)$

$$\begin{aligned} \mathrm{E}\left(f(Z)\right) &= \mathrm{E}\left(q(Z)H_n(Z)\right) + \mathrm{E}\left(r(Z)\right) \\ &= \mathrm{E}\left(q(Z)\,\delta^n 1\right) + \mathrm{E}\left(r(Z)\right) \\ &= \mathrm{E}\left(d^n q(Z)\right) + \mathrm{E}\left(r(Z)\right) = \mathrm{E}\left(r(Z)\right) \qquad \text{iff } \mathrm{E}(d^n q(Z)) = 0. \end{aligned}$$

- Note that $d^n q(Z) = 0$ if and only if q has degree smaller than n and this is only if f has degree smaller or equal to $2n-1$. But also

$$\mathrm{E}\left(d^n q(Z)\right) = \mathrm{E}\left(d^n \sum_{i=0}^{\infty} c_i(q) H_i\right) = \langle H_n, \sum_{i=0}^{\infty} c_i(q) H_i \rangle = n! c_n(q) = 0$$

iff $c_n(q) = 0$.

Gaussian quadrature

For $k = 1, \ldots, n$ and $x_1, \ldots, x_n \in \mathbb{R}$ pairwise distinct, define the Lagrange polynomials

$$l_k(x) = \prod_{i:i\neq k} \frac{x - x_i}{x_k - x_i}$$

- These are indicator polynomial functions of degree $n-1$, namely $l_k(x_i) = \delta_{ik}$,
- and form a $\mathbb{R}$-vector space basis of the set of polynomials of degree at most $(n-1)$, $\mathbb{P}_{n-1}$.
- Hence if r has degree smaller than n then $r(x) = \sum_{k=1}^{n} r(x_k) l_k(x)$
- and for $\lambda_k = \mathrm{E}(l_k(Z))$ by linearity

$$\mathrm{E}(r(Z)) = \sum_{k=1}^{n} r(x_k)\,\mathrm{E}(l_k(Z)) = \sum_{k=1}^{n} r(x_k)\lambda_k$$

- Putting all together, on $\mathcal{D}_n = \{x : H_n(x) = 0\} = \{x_1, \dots, x_n\}$ and for f polynomial of degree at most $(2n-1)$ or s.t. $c_n(\frac{f-r}{H_n}) = 0$,

$$\begin{aligned} \mathrm{E}\left(f(Z)\right) = \mathrm{E}\left(r(Z)\right) &= \sum_{k=1}^{n} r(x_k)\,\mathrm{E}\left(l_k(Z)\right) \\ &= \sum_{k=1}^{n} f(x_k)\lambda_k \\ &= \mathrm{E_n}\left(f(X)\right) \end{aligned}$$

where $\mathrm{P_n}\left(X = x_k\right) = \mathrm{E}\left(l_k(Z)\right) = \lambda_k$ is a probability on $\mathcal{D}$.

- In general

$$\mathrm{E}\left(f(Z)\right) = \sum_{k=1}^{n} f(x_k)\,\mathrm{E}\left(l_k(Z)\right) + n!c_n(q).$$

An application: identification of Fourier coefficients

For $f(x) = \sum_{k=0}^{N} c_k(f) H_k(x)$, then

- $\mathrm{E}(f(Z)) = c_0(f)$
- if $c_n((f-r)/H_n) = 0$ e.g. if $N \leq 2n-1$ then

$$c_0(f) = \sum_{H_n(x_k)=0} f(x_k)\lambda_k.$$

- If $c_n((f\,H_i - r)/H_n) = 0$ e.g. for all i such that $N + i \leq 2n-1$

$$\sum_{H_n(x_k)=0} f(x_k) H_i(x_k) \lambda_k = \mathrm{E}(f(Z)H_i(Z)) = i! c_i(f),$$

- in particular if $\deg f = n-1$ then all coefficients can be computed exactly.
- In general

$$\sum_{H_n(x_k)=0} f(x_k) H_i(x_k) \lambda_k = \sum_{H_n(x_k)=0} \mathrm{NF}(f(x_k)H_i(x_k))\lambda_k$$
$$= \mathrm{E}(\mathrm{NF}(f(Z)H_i(Z))) = i! c_i(\mathrm{NF}(f)).$$

III. Algebraic computation of the weights λ_k

Theorem

Let λ be the polynomial of degree $n-1$ such that $\lambda(x_k) = \lambda_k$ then

$$\lambda(x) H_{n-1}^2(x) = \frac{(n-1)!}{n} \qquad \text{on } H_n(x) = 0.$$

- E.g. for $n = 3$

$$\begin{cases} 0 &= H_3(x) = x^3 - 3x \\ 2/3 &= \lambda(x) H_2^2 = (\theta_0 + \theta_1 x + \theta_2 x^2)(x^2-1)^2 \end{cases}$$

  reduce degree using $x^3 = 3x$ and equate coefficients to obtain

$$\lambda(x) = \frac{2}{3} - \frac{x^2}{6}$$

  Evaluate to find $\lambda_{-\sqrt{3}} = \lambda(-\sqrt{3}) = \frac{1}{6} = \lambda_{\sqrt{3}}$ and $\lambda_0 = \lambda(0) = \frac{2}{3}$.

- The roots of H_n, $n > 2$, are not in $\mathbb{Q}$. Computer algebra systems work with rational fields. Working with algebraic extensions of fields could be slow.
- Sometimes there is no need to compute explicitly the weights.

A CoCoA code for the weighing polynomial

```
N:=4;                                   -- number of nodes
Use R::=Q[w,h[1..(N-1)]], Elim(w);      -- setting up the ring
Eqs:=[h[2]-h[1]*h[1]+1];                -- the Hermite polynomials
For I:=3 To N-1 Do
    Append(Eqs,h[I]-h[1]*h[I-1]+(I-1)*h[I-2])
EndFor;
Append(Eqs,h[1]*h[N-1]-(N-1)*h[N-2]); --the nodes
Set Indentation;
Append(Eqs,N*w*h[N-1]^2-Fact(N-1));    --the weighing polynomial
J:=Ideal(Eqs); GB_J:=GBasis(J);         --the game
Last(GB_J);

3w + 1/4h[2] - 5/4                      --the result
-------------------------------
```

Hence, $w(x) = \frac{5-h[2]}{12} = \frac{6-x^2}{12}$ and as $h[4] = x^4 - 6x^2 + 3 = 0$ we get

x	$-\sqrt{3-\sqrt{6}}$	$-\sqrt{3 \pm \sqrt{6}}$	$\sqrt{3-\sqrt{6}}$	$\sqrt{3+\sqrt{6}}$
$w(x)$	$\frac{3+\sqrt{6}}{12}$	$\frac{3-\sqrt{6}}{12}$	$\frac{3+\sqrt{6}}{12}$	$\frac{31\sqrt{6}}{12}$

Proof

- For $\{\tilde{H}_n\}_n$ a sequence of normalised orthogonal polynomials, the Christoffel-Darboux formula recite

$$\sum_{k=0}^{n-1} \tilde{H}_k(x)\tilde{H}_k(t) = \sqrt{\beta_n}\frac{\tilde{H}_n(x)\tilde{H}_{n-1}(t) - \tilde{H}_{n-1}(x)\tilde{H}_n(t)}{x-t}$$

$$\sum_{k=0}^{n-1} \tilde{H}_k(t)^2 = \sqrt{\beta_n}\left(\tilde{H}_n'(t)\tilde{H}_{n-1}(t) - \tilde{H}_{n-1}'(t)\tilde{H}_n(t)\right)$$

where $\tilde{H}_{k+1}(t) = (t-\alpha_k)\tilde{H}_k(t) - \beta_k\tilde{H}_{k-1}(t) \qquad \tilde{H}_{-1}(t) = 0 \qquad \tilde{H}_0(t) = 1.$

- For $\tilde{H}_n = H_n/\sqrt{n!}$ and at points in $\mathcal{D}_n = \{x : H_n(x) = 0\}$ they become

$$\sum_{k=0}^{n-1} \tilde{H}_k(x_i)\tilde{H}_k(x_j) = 0 \text{ if } i \neq j \qquad \sum_{k=0}^{n-1} \tilde{H}_k(x_i)^2 = n\tilde{H}_{n-1}(x_i)^2$$

Hence for $\mathbb{H}_n = \left[\tilde{H}_j(x_i)\right]_{i=1,\dots,n;j=0,\dots,n-1}$

$$\mathbb{H}_n\mathbb{H}_n^t = n\, diag(\tilde{H}_{n-1}^2(x_i) : i = 1,\dots,n)$$

and $\mathbb{H}_n^{-1} = \mathbb{H}_n^t n^{-1} diag(\tilde{H}_{n-1}^{-2}(x_i) : i = 1,\dots,n)$.

- Let $f(x) = \sum_{j=0}^{n-1} c_j \tilde{H}_j(x)$ and $\underline{f} = \mathbb{H}_n \underline{c}$ where $\underline{f} = [f(x_i)]_{i=1,\dots,n}$ and $\underline{c} = [c_j]_j$. Furthermore

$$\underline{c} = \mathbb{H}_n^{-1} \underline{f} = \mathbb{H}_n^t n^{-1} diag(\tilde{H}_{n-1}^{-2}(x_i) : i = 1, \dots, n) \underline{f}$$
$$= \mathbb{H}_n^t n^{-1} diag(\tilde{H}_{n-1}^{-2}(x_i) f(x_i) : i = 1, \dots, n)$$
$$c_j = \frac{1}{n} \sum_{i=1}^{n} \tilde{H}_j(x_i) f(x_i) \tilde{H}_{n-1}^{-2}(x_i)$$

- For $f(x) = l_k(x)$ the kth Lagrange polynomial and using $l_k(x_i) = \delta_{ik}$ above

$$c_j = \frac{1}{n} \tilde{H}_j(x_k) \tilde{H}_{n-1}^{-2}(x_k)$$

- The expected value of $l_k(Z)$ is

$$\lambda_k = \mathrm{E}\left(l_k(Z)\right) = \sum_{j=0}^{n-1} c_j \, \mathrm{E}\left(\tilde{H}_j(x)\right) = c_0 = \frac{1}{n} \tilde{H}_{n-1}^{-2}(x_k)$$

IV. Fractions: $\mathcal{F} \subset \mathcal{D}_n$, $\#\mathcal{F} = m < n$

- Let $1_{\mathcal{F}}(x)$ be the polynomial of degree n such that $1_{\mathcal{F}}(x) = 1$ if $x \in \mathcal{F}$ and 0 if $x \in \mathcal{D}_n \setminus \mathcal{F}$ and

  let f be polynomial of degree at most $n-1$ or s.t. $c_n((f1_{\mathcal{F}} - r)/H_n) = 0$ and

  let $Z \sim \mathcal{N}(0,1)$.

  Then

$$\mathrm{E}((f1_{\mathcal{F}})(Z)) = \sum_{x_k \in \mathcal{F}} f(x_k)\lambda_k = \mathrm{E_n}\left(f(X)1_{\mathcal{F}}(X)\right)$$
$$= \mathrm{E_n}\left(f(X)|X \in \mathcal{F}\right)\mathrm{P_n}(X \in \mathcal{F})$$

  where $\mathrm{P_n}(X = x_k) = \lambda_k$.

- Let $\omega_{\mathcal{F}}(x) = \prod_{x_k \in \mathcal{F}} (x - x_k) = \sum_{i=0}^{m} c_i H_i(x)$[1] and

  note $l_k^{\mathcal{F}}(x) = \prod_{i \in \mathcal{F}, i \neq k} \frac{x - x_i}{x_k - x_i} = \mathrm{NF}(l_k(x), \mathrm{Ideal}(\omega_{\mathcal{F}}(x))$ are the Lagrange polynomials for $\mathcal{F}$.

  Let f be a polynomial of degree N and $f(x) = q(x)\omega_{\mathcal{F}}(x) + r(x)$ with $f(x_i) = r(x_i)$ on $\mathcal{F}$ and $r(x) = \sum_{x_k \in \mathcal{F}} f(x_k) l_k^{\mathcal{F}}(x)$.

  Let's write $q(x) = \sum_{j=0}^{N-m} b_j H_j(x)$.

  Then

$$\mathrm{E}\left(f(Z)\right) = \mathrm{E}\left(\sum_{j=0}^{N-m} b_j H_j(Z) \sum_{i=0}^{m} c_i H_i(Z)\right) + \mathrm{E}\left(r(Z)\right)$$

$$= b_0 c_0 + b_1 c_1 + \ldots + ((N-m) \wedge m)! b_{(N-m)\wedge m} c_{(N-m)\wedge m} + \sum_{x_k \in \mathcal{F}} f(x_k) \lambda_k^{\mathcal{F}}$$

  where $\lambda_k^{\mathcal{F}} = \mathrm{E}\left(\mathrm{NF}(l_k(x), \mathrm{Ideal}(\omega_{\mathcal{F}}(x)))\right)$.

---

[1] cf. the node polynomial from Gautschi.

V. Higher dimension

Theorem

Let $Z_1, \ldots, Z_d$ *i.i.d.* $\sim \mathcal{N}(0,1)$, $f \in \mathbb{R}[x_1, \ldots, x_d]$ *with* $\deg_{x_i} f \leq 2n_i - 1$ *for* $i = 1, \ldots, d$ *and*

$$\mathcal{D}_{n_1 \ldots n_d} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : H_{n_1}(x_1) = H_{n_2}(x_2) = \ldots = H_{n_d}(x_d) = 0\}.$$

Then

$$\mathrm{E}\left(f(Z_1, \ldots, Z_d)\right) = \sum_{(x_1, \ldots, x_d) \in \mathcal{D}_{n_1 \ldots n_d}} f(x_1, \ldots, x_d) \lambda_{x_1}^{n_1} \ldots \lambda_{x_d}^{n_d}$$

where $\lambda_{x_j}^{n_j} = \mathrm{E}\left(\lambda_{x_j}(Z_j)\right)$ *for* $x_j \in \mathcal{D}_{n_j}$.

- Can take other f e.g. for $f(x,y) = q_1(x,y)H_n(x) + q_2(x,y)H_m(y) + r(x,y)$ is needed that $\mathrm{E}\left(d_x^n q_1(Z_1, Z_2)\right)$ and $\langle H_m(Z_2), q_2(Z_1, Z_2)\rangle = 0$.
- For Z_1 and Z_2 dependent with known covariance then without changing degrees the previous applies.

An application

Consider $\mathcal{D}_{nn}$ and f a polynomial with $\deg_x f, \deg_y f < n$ then

$$f(x,y) = \sum_{i,j=0}^{n-1} c_{ij} H_i(x) H_j(y)$$

As $\deg_x(fH_k), \deg_y(fH_k) < 2n-1$ for all $k < n$, then

$$\begin{aligned} \mathrm{E}\left(f(Z_1,Z_2)H_k(Z_1)H_h(Z_2)\right) &= c_{hk}\delta_{ik}||H_k(Z_1)||^2\delta_{jh}||H_h(Z_2)||^2 \\ c_{kh} &= \frac{1}{k!h!}\sum_{(x,y)\in\mathcal{D}_{nn}} f(x,y)H_k(x)H_h(y)\lambda_x\lambda_y \end{aligned}$$

Note if f is the indicator function of a fraction $\mathcal{F} \subset \mathcal{D}_{nn}$ then

$$c_{kh} = \frac{1}{k!h!}\sum_{(x,y)\in\mathcal{F}} H_k(x)H_h(y)\lambda_x\lambda_y$$

Fraction: an example

- $$\begin{cases} g_1 = x^2 - y^2 & = H_2(x) - H_2(y) = 0 \\ g_2 = y^3 - 3y & = H_3(y) = 0 \\ g_3 = xy^2 - 3x & = H_1(x)\left(H_2(y) - 2H_0\right) = 0 \end{cases}$$
- For any f polynomial there exists unique $r \in \text{Span}\left(1, x, y, xy, y^2\right) = \text{Span}\left(H_0, H_1(x), H_1(y), H_1(x)H_1(y), H_2(y)\right)$ such that $f = \sum_{i=1}^{3} q_i g_i + r$.
- If

$$q_1(x,y) = a_o + a_1 H_1(x) + a_2 H_1(y) + a_3 H_1(x) H_1(y)$$
$$q_2(x,y) = \theta_1(x) + \theta_2(x) H_1(y) + \theta_3(x) H_2(y)$$
$$q_3(x,y) = a_4 + a_5 H_1(y)$$

- Then

$$\begin{aligned} \mathrm{E}(f(Z_1, Z_2)) &= \mathrm{E}(r(Z_1, Z_2)) \\ &= 2\frac{f(0,0)}{3} + \frac{f(\sqrt{3},\sqrt{3}) + f(\sqrt{3},-\sqrt{3}) + f(-\sqrt{3},\sqrt{3}) + f(-\sqrt{3},-\sqrt{3})}{12} \end{aligned}$$

Summary

Input: $\mathcal{F} \subset \{(x_1, \ldots, x_d) : H_{n_i}(x_i) = 0 \; i = 1, \ldots, d\}$
τ and f polynomial
Output: $\mathrm{E}(f(Z))$ with $Z \sim \mathcal{N}_n(0, I)$

1. Compute G, a τ-Gröbner basis of the design ideal of $\mathcal{F}$
2. Let $H = \{h = h_{a_1}(x_1) \ldots h_{a_d}(x_d) : \mathrm{LT}_\tau(h) \leq_\tau \mathrm{LT}_\tau(g)$ for all $g \in G\}$
(the Hermite basis of the linear space of monomials in the g's)
3. Write $g \in G$ in terms of Hermite polynomials
(change of linear basis from "monomials" to "Hermite")
4. Write $f = \sum_{g \in G} s_g g + r = \sum_{g \in G} s_g \sum_{h < g} g_h h + r$
5. Check if $\sum_{g \in G, h < g} \langle s_g g_h, h \rangle = 0$ for all $h = h_{a_1}(x_1) \ldots h_{a_d}(x_d) \in H$
(more often than not complicated linear combination of coefficients of f)
6. If YES then $\mathrm{E}(f(Z)) = \sum_{x \in \mathcal{F}} f(x) \lambda_x$
7. If NO then $\mathrm{E}(f(Z)) = \sum_{x \in \mathcal{F}} f(x) \lambda_x +$ complicated linear combination of coefficients of f

Notes:

- 2., 3. and 4. are linear operations
- Find $\mathcal{F}$ and f such that 5. holds
- Do the algorithm directly in Hermite polynomials but do not go linear in the H's.

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