Multidegree for bifiltered D-modules and hypergeometric systems

Rémi Arcadias
Tokyo Woman's Christian University

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Weyl algebra

$$D = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

with for any i, $\partial_i x_i = x_i \partial_i + 1$.

Definition

F-filtration: defined by the weights

x_1	...	x_n	∂_1	...	∂_n
0	...	0	1	...	1

$P = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha \partial^\beta$ belongs to $F_d(D)$ iff any $x^\alpha \partial^\beta$ has weight $\leq d$. $D = \cup_{d \in \mathbb{N}} F_d(D)$.

Definition

V-filtration (along the origin): defined by the weights

x_1	...	x_n	∂_1	...	∂_n
-1	...	-1	1	...	1

$D = \cup_{k \in \mathbb{Z}} V_k(D)$.

Bifiltered D-module

Definition

Bifiltration: $F_{d,k}(D) = F_d(D) \cap V_k(D)$.

For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, $D^r[\mathbf{n}][\mathbf{m}] = D^r$ with

$$F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) = \oplus_{i=1}^{r} F_{d-n_i,k-m_i}(D).$$

Definition

A bifiltered D-module M is a D-module $D^r[\mathbf{n}][\mathbf{m}]/N$ with the quotient bifiltration

$$F_{d,k}(M) = \frac{F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) + N}{N}.$$

Bifiltered free resolution

A bifiltered free resolution of a bifiltered D-module M is an exact sequence

$$0 \to D^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}] \to \cdots \to D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \to M \to 0$$

such that for any d, k, we have an exact sequence

$$0 \to F_{d,k}(D^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}]) \to \cdots$$

$$\to F_{d,k}(D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}]) \to F_{d,k}(M) \to 0.$$

How to use the Betti numbers r_i and the shifts $\mathbf{n}^{(i)}, \mathbf{m}^{(i)}$?

Example

$D = \mathbb{C}[x_1, x_2]\langle\partial_1, \partial_2\rangle$
Let I be the ideal generated by $\partial_1 - \partial_2$ and $x_1\partial_1 + x_2\partial_2$.
$M = D/I = M_A(0,0)$ with $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$.
Bifiltered free resolution:

$$0 \to D^1[2][1] \xrightarrow{\phi_2} D^2[1,1][1,0] \xrightarrow{\phi_1} D^1[0][0] \to M \to 0$$

with

- $\phi_1(e_1) = \partial_1 - \partial_2$
- $\phi_1(e_2) = x_1\partial_1 + x_2\partial_2$
- $\phi_2(1) = (x_1\partial_1 + x_2\partial_2)e_1 - (\partial_1 - \partial_2)e_2$

K-polynomial: example

$$0 \to D^1[2][1] \xrightarrow{\phi_2} D^2[1,1][1,0] \xrightarrow{\phi_1} D^1[0][0] \to M \to 0$$

- $K(D^1[0][0]; T_1, T_2) = T_1^0 T_2^0 = 1$
- $K(D^2[1,1][1,0]; T_1, T_2) = T_1^1 T_2^1 + T_1^1 T_2^0 = T_1 T_2 + T_1$
- $K(D^1[2][1]; T_1, T_2) = T_1^2 T_2^1 = T_1^2 T_2$
- Then $K(M; T_1, T_2) = 1 - (T_1 T_2 + T_1) + T_1^2 T_2$

K-polynomial: definition and invariance

$$0 \to D^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}] \to \cdots \to D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \to M \to 0$$

- $K(D^r[\mathbf{n}][\mathbf{m}]; T_1, T_2) = \sum_j T_1^{n_j} T_2^{m_j}$
- $K(M; T_1, T_2) = \sum_i (-1)^i K(D^{r_i}[\mathbf{n}^{(i)}][\mathbf{m}^{(i)}]; T_1, T_2)$

Proposition

$K(M; T_1, T_2) \in \mathbb{Z}[T_1, T_1^{-1}, T_2, T_2^{-1}]$ *does not depend on the bifiltered free resolution.*

But it depends on the bifiltration, i.e. the presentation of M.

Multidegree: example

$K(M; 1-T_1, 1-T_2) \in \mathbb{Z}[[T_1, T_2]]$.
Example: $K(M; T_1, T_2) = 1 - (T_1 T_2 + T_1) + T_1^2 T_2$

$$
\begin{aligned}
K(M; 1-T_1, 1-T_2) &= 1 - ((1-T_1)(1-T_2) + (1-T_1)) \\
&+ ((1-T_1)^2(1-T_2)) \\
&= (T_1^2 + T_1 T_2) - T_1^2 T_2
\end{aligned}
$$

Multidegree:

$$\mathcal{C}(M; T_1, T_2) = T_1^2 + T_1 T_2.$$

It is an element of $\mathbb{Z}[T_1, T_2]$, homogeneous of degree 2.

V-homogenization

Let $P = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha \partial^\beta \in D$.

Definition (V-homogenization)

- $\mathrm{ord}^V(P) = \max_{\alpha,\beta}(\sum_i \beta_i - \sum_i \alpha_i)$
- $H^V(P) = \sum_{\alpha,\beta} x^\alpha \partial^\beta \theta^{\mathrm{ord}^V(P) - (\sum \beta_i - \sum \alpha_i)} \in D[\theta]$.
- *Let I be an ideal of D. Let $H^V(I)$ be the ideal of $D[\theta]$ generated by $H^V(P), P \in I$.*
- *If $M = D/I$, then $\mathcal{R}_V(M) = D[\theta]/H^V(I)$.*

F-filtration on $D[\theta]$: θ has weight 0.

$$\mathrm{gr}^F(D[\theta]) \simeq \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n, \theta] \quad \text{bigraded by}$$

	x_1	...	x_n	∂_1	...	∂_n	θ
F	0	...	0	1	...	1	0
V	-1	...	-1	1	...	1	1

Multidegree: definition and invariance

Let $d = \operatorname{codim}(\operatorname{gr}^F(\mathcal{R}_V(M)))$.

Definition

The multidegree $\mathcal{C}(M; T_1, T_2)$ is the sum of the terms whose degree equals d in the expansion of $K(M; 1 - T_1, 1 - T_2)$.

Theorem

The multidegree $\mathcal{C}(M; T_1, T_2)$ neither depends on the bifiltered free resolution nor on the bifiltration.

Outline of the proof:

- $\mathcal{C}(M; T_1, T_2) = \mathcal{C}(\operatorname{gr}^F(\mathcal{R}_V(M)); T_1, T_2)$ only depends on the cycle (maximal dimensional associated primes and multiplicities) associated with $\operatorname{gr}^F(\mathcal{R}_V(M))$.
- This cycle does not depend on the bifiltration, using a proof by Laurent [2].

Rémi Arcadias
Tokyo Woman's Christian University

Hypergeometric systems

Let $A = (a_{i,j})$ be a $d \times n$ integer matrix of rank d.
Assume

- The columns $a_1, \ldots, a_n$ generate $\mathbb{Z}^d$ over $\mathbb{Z}$
- $a_1, \ldots, a_n$ lie in a single halfspace.

Let I_A be the ideal of $\mathbb{C}[\partial_1, \ldots, \partial_n]$ generated by the elements $\partial^u - \partial^v$ with $u, v \in \mathbb{N}^n$ such that $Au = Av$.
Let $\beta_1, \ldots, \beta_d \in \mathbb{C}$ and $H_A(\beta)$ be the ideal of D generated by I_A and the elements

$$\sum_j a_{i,j} x_j \partial_j - \beta_i,$$

for $i = 1, \ldots, d$.
Hypergeometric system (Gel'fand-Zelevinsky-Kapranov):

$$M_A(\beta) = D/H_A(\beta).$$

Homogeneous case

Assumption: The columns $a_1, \ldots, a_n$ lie in a single hyperplane.
$\mathrm{vol}(A)$ = volume of the convex hull in $\mathbb{R}^d$ of the set $\{0, a_1, \ldots, a_n\}$ (with $\mathrm{vol}([0,1]^d) = d!$).

Theorem

If moreover $\mathbb{C}[\partial]/I_A$ is Cohen-Macaulay, then for any $\beta \in \mathbb{C}^d$ we have

- $\mathrm{codim}(\mathrm{gr}^F(\mathcal{R}_V(M_A(\beta)))) = n$
- $\mathcal{C}(M_A(\beta); T_1, T_2) = \mathrm{vol}(A). \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j}$.

Example: Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.
I_A is generated by $\partial_1 \partial_3 - \partial_2^2$.
For all β,

$$\mathcal{C}(M_A(\beta); T_1, T_2) = 2T_1^3 + 2T_1^2 T_2.$$

Proof of the theorem

- $M_A(\beta)$ is V-homogeneous, then $\mathcal{R}_V(M_A(\beta)) = D[\theta]/H_A(\beta)$.
- Then $\mathrm{codim}(\mathrm{gr}^F(\mathcal{R}_V M_A(\beta))) = n$.
- Let $(Ax\xi)_i = \sum_j a_{i,j}x_j\xi_j \in \mathbb{C}[x,\xi] = \mathrm{gr}^F(D)$. By Saito-Sturmfels-Takayama, $(Ax\xi)_1, \dots, (Ax\xi)_d$ is a regular sequence in $\mathrm{gr}^F(D/I_A) = \mathbb{C}[x,\xi]/I_A$.
- Then $\mathrm{gr}^F(\mathcal{R}_V M_A(\beta)) = \mathbb{C}[x,\xi,\theta]/(I_A + (Ax\xi))$.
- Moreover $\mathcal{C}(M_A(\beta); T_1, T_2) = \mathcal{C}(\mathbb{C}[\xi]/I_A; T_1, T_2).T_1^d$.
- $\mathcal{C}(\mathbb{C}[\xi]/I_A; T_1, T_2) = \mathcal{C}(\mathbb{C}[\xi]/I_A; T)_{T=T_1+T_2}$.
- $\mathcal{C}(\mathbb{C}[\xi]/I_A; T) = \deg(\mathbb{C}[\xi]/I_A)T^{n-d}$.
- $\deg(\mathbb{C}[\xi]/I_A) = \mathrm{vol}(A)$.
- Finally, $\mathcal{C}(M_A(\beta); T_1, T_2) = \mathrm{vol}(A).(T_1 + T_2)^{n-d}T_1^d$.

Homogeneous case: non Cohen-Macaulay example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

I_A is generated by
$\partial_2\partial_4^2 - \partial_3, \quad \partial_1\partial_4 - \partial_2\partial_3, \quad \partial_1\partial_3^2 - \partial_2^2\partial_4, \quad \partial_1^2\partial_3 - \partial_2^3.$
For $(\beta_1, \beta_2) \neq (1, 2)$, we have

$$\mathcal{C}(M_A(\beta); T_1, T_2) = 4T_1^4 + 8T_1^3T_2 + 4T_1^2T_2^2.$$

For $(\beta_1, \beta_2) = (1, 2)$, we have

$$\mathcal{C}(M_A(\beta); T_1, T_2) = 5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.$$

Inhomogeneous case

Let $I \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ be an ideal.
$H(I) \subset \mathbb{C}[\partial_1, \ldots, \partial_n, h]$ is the F-homogenization of I.

Theorem

Assume that $\mathbb{C}[\partial, h]/H(I_A)$ is Cohen-Macaulay. Then for generic β, we have

- $\operatorname{codim}(\operatorname{gr}^F(\mathcal{R}_V(M_A(\beta)))) = n$
- $\mathcal{C}(M_A(\beta); T_1, T_2) = \operatorname{vol}(A). \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j}$.

Example:

Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 4 & 3 & 2 \end{pmatrix}$.
I_A is generated by $\partial_1^7 \partial_3^4 - \partial_2^{12}$.
For any β,

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 12 T_1^3 + 12 T_1^2 T_2.$$

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