Geometry of Blaschke products and bicentric polygons

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Part I

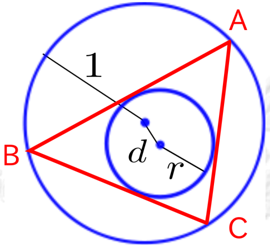
Chapple-Euler formula

Bicentric polygon is a polygon which has both inscribed circle and circumscribed circle.

Theorem (Chapple-Euler (1746, 1765))

The distance d between the circuncenter and incenter of a triangle is given by $d^2 = R(R - 2r)$, where R and r are the circunradius and inradius, respectively. In particular, if circumscribed circle is the unit circle, then the distance is given by

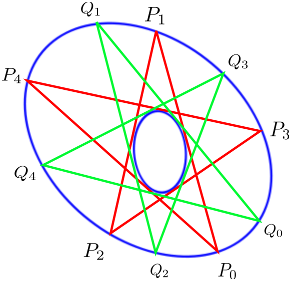
$$d^2 = 1 - 2r.$$

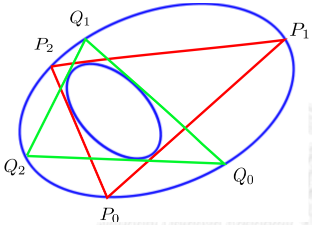


Poncelet's porism

Theorem (Poncelet (1822))

As shown in the following figure, ellipse E_2 is in the inside of ellipse E_1. If polygonal line inscribed in E_1 and simultaneously tangent to E_2 closes for one particular position of the start point P_0 on E_1, then it closes for every position of E_1.





Blaschke product

A Blaschke product of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

The Blaschke product is an holomorphic function on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and maps $\mathbb{D}$ to itself.

Let

$$B(z) = z \prod_{k=1}^{d-1} \frac{z - a_k}{1 - \overline{a_k} z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

- Every point of $\partial\mathbb{D}$ has exactly different d preimages on $\partial\mathbb{D}$.
- Let $z_1, \cdots, z_d$ be d preimages of $\lambda \in \partial\mathbb{D}$, and $F(z) = \frac{B(z)/z}{B(z)-\lambda} = \sum_{k=1}^{d} \frac{m_k}{z-z_k}$. Then, m_j satisfy

$$\sum_{k=1}^{d} m_k = 1 \quad \text{and} \quad 0 < m_k < 1 \ (k = 1, \cdots, d).$$

The Results of Daepp, Gorkin and Mortini

Theorem (Daepp, Gorkin and Mortini (2002))

Let $B(z) = z\dfrac{z-a}{1-\overline{a}z}$ $(a \neq 0,\ a \in \mathbb{D})$. For $\lambda \in \partial\mathbb{D}$, let $z_1,\ z_2$ be the two distinct points satisfying $B(z_1) = B(z_2) = \lambda$. Then the line joining z_1 and z_2 passes through the point a.

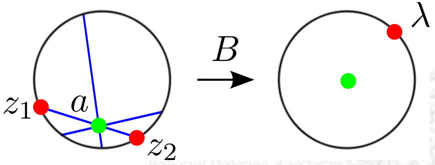
Conversely, the two intersection points ζ_1, ζ_2 of any line through the point a and $\partial\mathbb{D}$ satisfy $B(\zeta_1) = B(\zeta_2)$.

This theorem holds for more general Blaschke product of degree 2,

$$B(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}\cdot\frac{z-b}{1-\overline{b}z}.$$

In this case the fixed point is given by

$$\frac{a+b-ab\overline{(a+b)}}{1-ab\overline{ab}}.$$



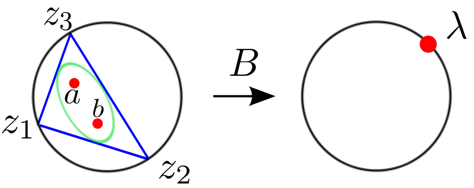
Theorem (Daepp, Gorkin and Mortini (2002))

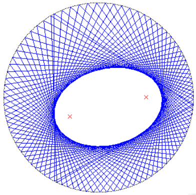
Let $B(z) = z\dfrac{z-a}{1-\overline{a}z}\cdot\dfrac{z-b}{1-\overline{b}z}$ $(a, b \neq 0,\ a \neq b,\ a, b \in \mathbb{D})$, and $z_1,\ z_2,\ z_3$ be the three distinct preimages of $\lambda \in \partial\mathbb{D}$ by B. Then, the lines joining z_k, z_ℓ $(k \neq \ell)$ is tangent to the ellipse

$$E: \quad |z-a| + |z-b| = |1-\overline{a}b|.$$

Conversely, each point of E is the point of tangency E of a line that passes through two distinct points ζ_1, ζ_2 on $\partial\mathbb{D}$ for which

$$B(\zeta_1) = B(\zeta_2).$$



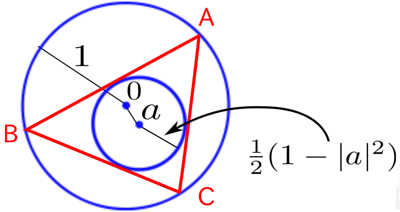


An another proof of Chapple-Euler formula

Corollary (multiple zeros)

Let $B(z) = z\left(\dfrac{z-a}{1-\overline{a}z}\right)^2$ $(a \neq 0,\ a \in \mathbb{D})$, and z_1, z_2, z_3 are preimages of $\lambda \in \partial\mathbb{D}$ by B.
Then, the line joining z_k, z_ℓ $(k \neq \ell)$ is tangent to the circle

$$C: \quad |z-a| = \frac{1}{2}(1-|a|^2).$$



Lemma

For three mutually distinct points z_1, z_2, z_3 on $\partial\mathbb{D}$, there exist a point $\lambda \in \partial\mathbb{D}$ and a point $a \in \mathbb{D}$ such that

$$B(z_1) = B(z_2) = B(z_3) = \lambda, \quad \text{where} \quad B(z) = z\Big(\frac{z-a}{1-\bar{a}z}\Big)^2.$$

Proof. Set

$$\lambda = z_1 z_2 z_3, \quad a = \frac{|z_1 - z_2|z_3 + |z_2 - z_3|z_1 + |z_3 - z_1|z_2}{|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|}.$$

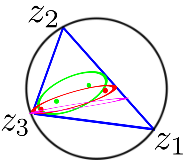
Then $B(z_1) = B(z_2) = B(z_3) = \lambda$. □

Corollary and Lemma gives an another proof of Chapple-Euler formula (both necessity and sufficiency !)

An inscribed ellipse and the circumscribed circle

Remark:

- There are many inscribed ellipses for a triangle $z_1 z_2 z_3$.
- When the point a tends to the line joining z_1, z_2, the point b converges to the point z_3.



Lemma

For three mutually distinct points z_1, z_2, z_3 on $\partial\mathbb{D}$ and every point a on the interior of triangle $z_1 z_2 z_3$, there exist a point $\lambda \in \partial\mathbb{D}$ and a Blaschke product B of degree 3 with zeros at the origin and a such that

$$B(z_1) = B(z_2) = B(z_3) = \lambda$$

Proof. Set $\lambda = z_1 z_2 z_3$, and $b = \frac{1}{1-|a|^2}(z_1 + z_2 + z_3 - \overline{a}\lambda(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}) - a + \overline{a}^2\lambda)$. Then we have

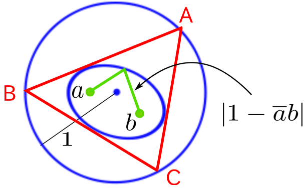
$$B(z) = z \cdot \frac{z-a}{1-\overline{a}z} \cdot \frac{z-b}{1-\overline{b}z}.$$

□

An extension of Chapple-Euler formula

Proposition

For any triangle that is inscribed in the unit circle, an ellipse is inscribed in the triangle if and only if the ellipse is associated with a Blaschke product of degree 3.



Theorem (An extension of Chapple-Euler)

Let ABC be the triangle with vertices at $A(\alpha)$, $B(\beta)$, $C(\gamma) \in \partial\mathbb{D}$, and a a one of the focus of the ellipse inscribed in the triangle ABC. Then, the other focus b is uniquely determined by

$$b = \frac{1}{1-|a|^2}\left(\alpha+\beta+\gamma-\overline{a}\lambda\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-a+\overline{a}^2\lambda\right),$$

and the sum of the distance between two focus points a, b and each point on the ellipse is given by $|1-\overline{a}b|$.

Conversely, for each pair of points $a,\ b \in \mathbb{D}$, there is a triangle which is inscribed in the unit circle and inscribes the ellipse $|z-a|+|z-b| = |1-\overline{a}b|$.

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