An algorithm of computing inhomogeneous differential equations for definite integrals

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- A new algorithm to compute inhomogeneous differential equations for definite integrals.
- Our algorithm is based on integration algorithm for D-modules, i.e. Gröbner basis methods in D.
- Example

$$F(x) = \int_a^b e^{-t-xt^3} dt$$

$$(27x^3\partial_x^2+54x^2\partial_x+6x+1)\cdot F(x) = -[(9t^4x^2-3t^2x-6tx+1)e^{-t-xt^3}]_{t=a}^{t=b}$$

- The Almkvist-Zeilberger algorithm, The Chyzak algorithm, The Oaku-Shiraki-Takayama algorithm.

Notations

- $\partial_i = \frac{\partial}{\partial x_i}$ differential operator of x_i
- Weyl Algebra in n variables $x_1, \cdots, x_n$

$$D = \mathrm{C}\langle x_1, \cdots, x_m, x_{m+1}, \cdots, x_n, \partial_1, \cdots, \partial_m, \partial_{m+1}, \cdots, \partial_n \rangle,$$

- Weyl Algebra in $n - m$ variables $x_{m+1}, \cdots, x_n$ $(m \leq n)$

$$D' = \mathrm{C}\langle x_{m+1}, \cdots, x_n, \partial_{m+1}, \cdots, \partial_n \rangle$$

D' is a subring of D.

- Commutative relations

$$\partial_i x_i = x_i \partial_i + 1, \quad x_i \partial_j = \partial_j x_i \ (i \neq j)$$

$$x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_i \partial_j$$

Integration ideal

Definition

The integration ideal of a left D-ideal I with respect to the variables $x_1, \cdots, x_m$ is the left D'-ideal

$$J = (I + \partial_1 D + \cdots + \partial_m D) \cap D'.$$

- $I + \partial_1 D + \cdots + \partial_m D$ is not a left D-ideal.
- To compute an integration ideal, we cannot simply apply the elimination method by the Gröbner basis in D.

$((I + D\partial_1 + \cdots + D\partial_m) \cap D')$

Integration algorithm for D-modules

Algorithm (Integration algorithm for D-modules (Oaku))

Input: Generators of holonomic left D-ideal I

Output : Generators of integration ideal of I with respect to $x_1, \ldots, x_m$

- Set a weight vector $w = (w_1, \cdots, w_m, 0, \cdots, 0) \quad (w_i > 0)$, where w_i means the weight of ∂_i and $-w_i$ means the weight of x_i.
- Compute the Gröbner basis with respect to a monomial order $<_{(-w,w)}$ in D.
- Compute generic b-function $b(s)$ of a holonomic D-ideal with repect to the weight vector w.
- Compute the left D'-module $(D')^r/M$ which is isomorphic to the integration module $D/(I + \partial_1 D + \cdots + \partial_m D)$. Here, r is detemined by the maximal non-integer root of $b(s) = 0$.
- Compute the Gröbner basis of M in the left D'-module $(D')^r$.

Application of integration ideals

We consider a definite integral of $f(x_1, \ldots, x_n)$ with respect to x_1

$$F(x_2, \ldots, x_n) = \int_a^b f(x_1, \cdots, x_n) dx_1.$$

1. $I = \mathrm{Ann}_D f := \{P \in D \mid P \cdot f = 0\}$
2. The integration ideal of I with respect to x_1 is
$$J = (I + \partial_1 D) \cap D' \quad D' = \mathbf{C}\langle x_2, \cdots, x_n, \partial_2, \cdots, \partial_n \rangle.$$
3. Take an element $P \in J$. There exists $P_0 \in I$ and $P_1 \in D$ such that
$$P = P_0 + \partial_1 P_1.$$
4. Apply P to the integral F.
$$P \cdot F(x_2, \cdots, x_n) = \int_a^b P \cdot f dx_1 = \int_a^b (P_0 + \partial_1 P_1) \cdot f dx_1$$
$$= \int_a^b \partial_1 P_1 \cdot f dx_1 = [P_1 \cdot f]_{x_1=a}^{x_1=b}$$

Example

$F(x) = \int_{-\infty}^{\infty} e^{-t^4-xt^3} dt$ (Oaku's text book "D-module and computer math" Ex5.19.)

1. The integrand $f(t,x) = e^{-t^4-xt^3}$ is annihilated by the holonomic ideal
$$I = \langle \partial_t + 4t^3 + 3xt^2, \partial_x + t^3 \rangle.$$
2. The integration ideal of I with respect to t is
$$J = \langle P, Q \rangle = \langle 64x^2\partial_x^3 - (27x^5 + 128x)\partial_x^2 - (81x^4 - 128)\partial_x - 15x^3,$$
$$64\partial_x^4 - 27x^3\partial_x - 216x^2\partial_x - 399x\partial_x - 45\rangle.$$
3. The operator P is $P = P_0 + \partial_t P_1 \quad (P_0 \in I, P_1 \in D)$.
4. We apply the operator P to the integral F

$$P \cdot \int_{-\infty}^{\infty} e^{-t^4-xt^3} dt = \int_{-\infty}^{\infty} P \cdot e^{-t^4-xt^3} dt = \int_{-\infty}^{\infty} (\partial_t P_1) \cdot e^{-t^4-xt^3} dt$$
$$= \int_{-\infty}^{\infty} \partial_t (P_1 \cdot e^{-t^4-xt^3}) dt = \left[P_1 \cdot e^{-t^4-xt^3} \right]_{t=-\infty}^{t=\infty} = 0$$

Our algorithm 1

Theorem-Algorithm (Computing inhomogeneous parts of an integration ideal)

Let J be the integration ideal of a holonomic left D-ideal I with respect to $x_1, \cdots, x_m$.

$$J = (I + \partial_1 D + \cdots \partial_m D) \cap D'$$

Then, for any $P \in J$, we can compute $P_0 \in I, P_1, \cdots, P_m \in D$ such that

$$P = P_0 + \partial_1 P_1 + \cdots + \partial_m P_m$$

Example

$$F(x) = \int_0^\infty e^{-t^4 - xt^3} dt$$

1. Integration ideal J

$$J = \langle P, Q \rangle = \langle 64x^2\partial_x^3 - (27x^5 + 128x)\partial_x^2 - (81x^4 - 128)\partial_x - 15x^3, \\ 64\partial_x^4 - 27x^3\partial_x - 216x^2\partial_x - 399x\partial_x - 45\rangle.$$

2. Compute inhomogeneous parts P_1 of P by using our algorithm. ($P = P_0 + \partial_t P_1, (P_0 \in I, P_1 \in D)$)

$$P_1 = \frac{1}{8}\{81x^5\partial_x - 27x^4 t\partial_t - x^2(20\partial_t^2 - 192\partial_x\partial_t + 576\partial_x^2) \\ - x(112\partial_t - 704\partial_x) - 256\}$$

3. Apply the operator P to the integral F.

$$P \cdot \int_0^\infty e^{-t^4 - xt^3} dt = \int_0^\infty \partial_t(P_1 \cdot e^{-t^4 - xt^3})dt = \left[P_1 \cdot e^{-t^4 - xt^3}\right]_{t=0}^{t=\infty}$$

$$= \left[(16x^2t^6 - 12x^3t^5 + 9x^4t^4 + 32xt^3 - 15x^3t + 32)e^{-t^4 - xt^3}\right]_{t=0}^{t=\infty}$$

$$= -32$$

Our algorithms 2

Theorem-Algorithm

We set

$$F(x_2, \ldots, x_n) = \int_a^b f(x) dx_1.$$

We can obtain holonomic inhomogeneous differential equations for the integral F from a holonomic ideal annihilating the integrand f.

Theorem-Algorithm

We set

$$F(x_{m+1}, \ldots, x_n) = \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(x) dx_1 \cdots dx_m.$$

We can obtain inhomogeneous differential equations for the integral F from a holonomic ideal annihilating the integrand f.

Summary

1. Algorithm computing inhomogeneous parts of an integration ideal
2. Algorithm computing inhomogeneous differential equations for a definite integral
3. Implementation of these algorithms to computer algebra system Risa/Asir (nk_restriction.rr)

- Reference
  - H.Nakayama, K.Nishiyama: `nk_restriction.rr`, `http://www.math.kobe-u.ac.jp/~nakayama/nk_restriction.rr`
  - H. Nakayama, K. Nishiyama: An algorithm of computing inhomogeneous differential equations for definite integrals, `arXiv:1005.3417`